Lecture 5. Lie Groups

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 5. Some Lie Groups and their personalities.

Let's recap our successful program from Lectures 1-4 for classifying the finite dimensional representations of finite groups:

(a) Average the ordinary dot product on a G-module to obtain an invariant dot (or Hermitian) product to prove *complete irreducibility*: every G-module is a direct sum of irreducible G-modules.

(b) Locate all the irreducible representations as G-submodules of a **single** G-module, namely the (left) regular representation $\mathbb{C}[G]$.

We're going to generalize this program to continuous (Lie) groups and algebraic groups. Before we worry about this, let's consider the cast of characters.

General Linear Groups. The multiplication map:

$$m : \mathrm{GL}(n,k) \times \mathrm{GL}(n,k) \to \mathrm{GL}(n,k)$$

is a **polynomial** map that is linear in the variables on each side of the product, and quadratic over all. For example:

$$m(a,b) = ab$$

$$m\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}\right) = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{bmatrix}$$

The inverse map:

$$i : \mathrm{GL}(n,k) \to \mathrm{GL}(n,k)$$

is a **rational map**. The one and two-dimensional cases are:

$$i(a) = \frac{1}{a}$$

$$i\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\right) = \begin{bmatrix} a_{1,1}/(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) & -a_{2,1}/(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \\ -a_{1,2}/(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) & a_{2,2}/(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \end{bmatrix}$$

The general case is *Cramer's rule*: the entries of A^{-1} are

$$A^{-1} = \left(\frac{(-1)^{i+j}\det(M_{j,i})}{\det(A)}\right)$$

i.e. the ratios of the determinants of $n-1 \times n-1$ *minors* $M_{i,j}$ and the determinant of A. Notice that these are *homogeneous* rational functions of degree -1, reflecting the fact that if A is replaced by λA, then A^{-1} is replaced by $\lambda^{-1}A^{-1}$.

These are *algebraic* maps, well defined over any field k.

Now consider the action of $\mathrm{GL}(n,k)$ on itself by conjugation. We've already seen that semi-simple matrices in $\mathrm{GL}(n,k)$ are those with a diagonal matrix in the conjugacy class. In general:

Jordan Canonical Form. Within each conjugacy class of $\mathrm{GL}(n,\mathbb{C})$, there is a matrix in canonical form, consisting of *Jordan blocks*:

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \vdots & \lambda \end{bmatrix}$$

corresponding to the "maximal cyclically generated" subspaces. These are subspaces generated by a vector $\vec{v}$, in the sense that the vectors:

$$\vec{v}_1 = \vec{v}, \vec{v}_2 = A\vec{v}_1 - \lambda\vec{v}_1, ..., \vec{v}_m = A\vec{v}_{m-1} - \lambda\vec{v}_{m-1}$$

span the space, and $\vec{v}_m$ is the only eigenvector.

Special Linear Groups are defined by a polynomial equation:

$$\mathrm{SL}(n,k) = \{A \in \mathrm{GL}(n,k) \mid \det(A) = 1\}$$

Notice that $\mathrm{SL}(2,k)$ has k^* as a subgroup:

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$

so it is *unbounded* when $k = \mathbb{R}$ or $\mathbb{C}$. This subgroup is not normal! When we conjugate it we get loads of "one-parameter" subgroups:

$$t \mapsto A \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} A^{-1}$$

The general linear group has the diagonal *torus* $T_n = (k^*)^n$ as a subgroup (as well as all its conjugate subgroups), and $\mathrm{SL}(n,k)$ contains the subtorus $T_{n-1} = \{(t_1, ., ..., t_n) \mid \prod_{i=1}^n t_i = 1\} \subset T_n$.

Projective Linear Groups are defined as the *quotient*

$$\mathrm{PGL}(n,k) = \mathrm{GL}(n,k)/k^*$$

by the *normal* subgroup consisting of multiples of the identity matrix. In this group, the natural torus is the *quotient* torus T_n/T via the diagonal action of T on T^n. Notice that the kernel of the map $\mathrm{SL}(n,k) \to \mathrm{PGL}(n,k)$ is the group C_n of roots of unity (times id). This is the center of the special linear group. Projective linear groups are not projective varieties...quite the opposite, they are affine, but they are the natural groups to act on projective space by automorphisms.

Orthogonal Groups. With respect to the ordinary dot product:

Definition 5.1. A matrix $A \in \mathrm{GL}(n, k)$ is *orthogonal* if

$$(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w} \text{ for all } \vec{v}, \vec{w} \in \mathbb{R}^n$$

The *orthogonal group* $O(n, k)$ is the group of orthogonal matrices. The *special orthogonal group* $\mathrm{SO}(n, k) \subset O(n, k)$ is the normal subgroup of orthogonal matrices of determinant 1.

Proposition 5.1. (a) $A \in O(n, k)$ if and only if

$$A(\vec{v}) \cdot A(\vec{v}) = \vec{v} \cdot \vec{v} \text{ for all } \vec{v} \in k^n$$

(b) A is orthogonal if and only if $A^T \cdot A = \mathrm{id}$.

(c) $\det(A) = 1$ or -1 if $A \in O(n, k)$.

Proof. One direction of (a) is obvious. As for the other, consider:

$$A(\vec{v} + \vec{w}) \cdot A(\vec{v} + \vec{w}) = A(\vec{v}) \cdot A(\vec{v}) + 2A(\vec{v}) \cdot A(\vec{w}) + A(\vec{w}) \cdot A(\vec{w})$$

and $(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}$. If we assume that all "square terms" are equal, then so are the cross terms.

The dot product $\vec{v} \cdot \vec{w}$ is the matrix product $(\vec{v})^T \cdot (\vec{w})$, where vectors are regarded as one-column matrices. A is orthogonal if:

$$(A\vec{v})^T \cdot (A\vec{w}) = (\vec{v})^T (A^T A)\vec{w} = (\vec{v}^T) \cdot (\vec{w})$$

for all $\vec{v}, \vec{w}$. Applying this to all pairs of basis vectors e_i, e_j gives (b). Then $\det(A)^2 = 1$, which gives (c).

Let us focus on the case $k = \mathbb{R}$.

Proposition 5.2. (a) The columns of $A \in O(n, \mathbb{R})$ are perpendicular unit vectors, i.e. an orthonormal basis for $\mathbb{R}^n$. Likewise for the rows.

(b) $O(n, \mathbb{R})$ are the symmetries of the unit sphere in $\mathbb{R}^n$.

(c) The stabilizer of a point of the sphere is isomorphic to $O(n-1, \mathbb{R})$.

Proof. (a) is an immediate consequence of Proposition 5.1(b), and (b) is an immediate consequence of Proposition 5.1(a). For (c), since all stabilizers are isomorphic, consider the stabilizer of the "north pole" $e_n \in S^{n-1}$. Then by (a) any stabilizer of e_n is a matrix of the form:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & 0 \end{bmatrix}$$

and again by (a) the submatrix of indeterminates is orthogonal! $\square$

Example 5.1. (a) Permutation matrices are orthogonal.

(b) We've seen $O(2,\mathbb{R})$ in Lecture 2. Namely,

$$O(2,\mathbb{R}) = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \right\} \sqcup \left\{ \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \right\}$$

are "rotations" and "reflections" which have determinant 1 and -1, respectively. The first column of each is an arbitrary point of the unit circle, and the second is a unit vector orthogonal to the first.

(c) The set of orthogonal matrices is a "tower of sphere bundles:"

$$O(n,\mathbb{R}) = X_n \to X_{n-1} \to \cdots \to X_1 = S^{n-1}$$

where X_i is the space of i ordered orthonormal vectors $\vec{u}_1, ..., \vec{u}_i \in \mathbb{R}^n$ (the first i columns of an orthogonal matrix). The map $X_{i+1} \to X_i$ forgets $\vec{u}_{i+1}$, which (given the first i vectors $\vec{u}_1, .., \vec{u}_i$) was free to be any point of the unit sphere $S^{n-i} \subset \langle \vec{u}_1, ..., \vec{u}_i \rangle^\perp$. In particular, the very last vector $\vec{u}_n$ is $\pm 1 \in \langle \vec{u}_1, ..., \vec{u}_{n-1} \rangle^\perp$ which produce matrices of determinant 1 or -1. In particular, $\mathrm{SO}(n,\mathbb{R})$ maps bijectively to X_{n-1}.

Example 5.2. The above discussion exhibits $\mathrm{SO}(3,\mathbb{R})$ as a bundle of circles over S^2, as does the action of $\mathrm{SO}(3,\mathbb{R})$ on the sphere. But there is another way to visualize $\mathrm{SO}(3,\mathbb{R})$:

Since $A \in \mathrm{SO}(3,\mathbb{R})$ preserves length, every eigenvalue of A is a real (or complex) number of length 1. From this and $\det(A) = 1$, it follows that $\lambda = 1$ is an eigenvalue for A with a one-dimensional eigenspace (unless $A = \mathrm{id}$). This eigenspace is the "axis" for a rotation in the perpendicular plane by an angle θ between $-\pi$ and π. In other words, every symmetry of the sphere is a rotation around an axis (obvious?). We will revisit this a little bit later.

Next, consider orthogonal matrices over $\mathbb{C}$.

The ordinary dot product is **not** the Hermitian product. Notice that $\vec{v} \cdot \vec{v}$ is **not** always positive, so this is not the square of a length. Nevertheless, we can analyze these matrices as we did in the real case.

With respect to the dot product:

• The columns (and rows) of an orthogonal matrix satisfy $\vec{v}_i \cdot \vec{v}_i = 1$ and $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$.

• The orthogonal group acts on the "complex sphere," i.e. the locus of vectors $\vec{u} \in \mathbb{C}^n$ that satisfy $\vec{u} \cdot \vec{u} = 1$.

• The orthogonal group is a "tower of complex spheres."

Example 5.3. $\mathrm{SO}(2,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$, as follows:

First of all, notice that a matrix in $\mathrm{SO}(2,\mathbb{C})$ is of the form:

$$A = \begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix} \text{ where } z_1^2 + z_2^2 = 1$$

In other words, as with $\mathrm{SO}(2,\mathbb{R})$, this **is** the first complex sphere. By analogy with rotations, here is the isomorphism:

$$z \mapsto \begin{bmatrix} (z+z^{-1})/2 & (z^{-1}-z)/2i \\ (z-z^{-1})/2i & (z+z^{-1})/2 \end{bmatrix}$$

Notice that when restricted to the unit complex numbers $e^{i\theta}$, this is **exactly** the isomorphism between the circle and $\mathrm{SO}(2,\mathbb{R})$.

Next, consider *unitary transformations* over the complex numbers.

Definition 5.2. A matrix $A \in \mathrm{GL}(n,\mathbb{C})$ is *unitary* if:

$$\langle A\vec{v}, A\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle \text{ for all } \vec{v}, \vec{w} \in \mathbb{C}^n$$

The *unitary group* $U(n)$ is the group of unitary matrices.

Example 5.4. $U(1) \subset \mathbb{C}^*$ is the unit circle!

The analogues of the results for real orthogonal groups hold:

- Unitary matrices are those that preserve length of (complex) vectors.
- A is unitary iff $A^T \cdot \overline{A} = \mathrm{id}$, which implies $\det(A) \cdot \overline{\det(A)} = 1$.
- The columns of a unitary matrix A are orthogonal unit vectors.
- The unitary groups are a tower of odd dimensional real spheres.

Definition 5.3. The *special unitary group* $\mathrm{SU}(n)$ is the kernel of

$$\det : U(n) \to U(1)$$

i.e. the unitary matrices of determinant 1.

Example 5.5. $\mathrm{SU}(2)$. A special unitary 2×2 matrix has the form:

$$\begin{bmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{bmatrix}$$

i.e. $\mathrm{SU}(2)$ is the unit sphere in $\mathbb{R}^4$ identified with:

$$\left\{ \begin{bmatrix} a_1 + ib_1 & -a_2 + ib_2 \\ a_2 + ib_2 & a_1 - ib_1 \end{bmatrix} \text{ such that } a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1 \right\}$$

but there is a better way to “see” this unit sphere:

The *quaternions* $\mathbb{H}$ are the set:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

with ordinary (vector) addition and vector multiplication defined by:

$$i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik$$

If we use vector notation $\vec{v} = v_1 i + v_2 j + v_3 k$, then:

$$(a + \vec{v})(b + \vec{w}) = (ab - \vec{v} \cdot \vec{w}) + (a\vec{w} + b\vec{v} + \vec{v} \times \vec{w})$$

where $\times$ is the cross product of vectors in $\mathbb{R}^3$. There is a quaternion conjugate and inverse. Let $q = a + \vec{v}$. Then:

$$\overline{q} = a - \vec{v},\ q\overline{q} = |q|^2 = a^2 + v_1^2 + v_2^2 + v_3^2,\ q^{-1} = \overline{q}/q\overline{q}$$

just as with the complex numbers. And as with the complex numbers: $S^3 = \{q \mid |q|^2 = 1\}$, i.e. the unit quaternions, are a group.

Proposition 5.3. *$SU(2)$ is isomorphic to the unit quaternions.*

More precisely, as above, an element of SU(2) is of the form:

$$A = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + b_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and the matrices multiply as $1, i, j, k$ (beware the multiple uses of i!). Thus the map $\phi : S^3 \to \mathrm{SU}(2)$ defined by:

$$\phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \phi(i) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \phi(j) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \phi(k) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

is the desired isomorphism (of groups).

Now consider the *action* of SU(2) on itself by *conjugation*:

Proposition 5.4. The conjugacy classes of elements in SU(2) are the "latitudes" of the sphere S^3, i.e. the unit quaternions of fixed real part.

Proof. If $q = a + \vec{v}$ and $r = b + \vec{w}$ are unit quaternions, then:

$$qrq^{-1} = b + (a^2 - |\vec{v}|^2)\vec{w} + 2a(\vec{v} \times \vec{w}) + 2(\vec{v} \cdot \vec{w})\vec{v}$$

(by brute force computation). This proves that the real part of a unit quaternion is fixed under conjugation. Alternatively, the real part of:

$$a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + b_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

is half of the trace, which is preserved under conjugation by matrices.

Next, suppose $r = b + \vec{w}$ and $r' = b + \vec{w}'$ are two unit quaternions in the same latitude. Then the unit quaternion $\vec{u}$ in the direction $\frac{1}{2}(\vec{w} + \vec{w}')$ (at the equator) conjugates r to r', since the effect of conjugating by $\vec{u}$ (by the brute force calculation) is to rotate the vector part by 180 degrees around the axis in the $\vec{u}$ direction, and that takes $\vec{w}$ to $\vec{w}'$. Thus conjugation is *transitive* on each latitude. □

Corollary 5.1. Via the conjugation action on the "equator:"

$$c : \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R});\ c(q)(0 + \vec{w}) = q(\vec{w})q^{-1}$$

is a surjective homomorphism of groups with $\ker(c) = \pm 1$.

Proof. The reader is invited to verify with brute force or, better, with elegance, that if $a + \vec{v} = q$ is a unit quaternion, then conjugation by q rotates the equator ($= S^2$) around the axis in the $\vec{v}$ direction by an angle ranging from 0 (at the north pole) to 2π (at the south pole) determined by a. Notice that this exhibits c as a *double cover* since the additional information of $\vec{v}$ determines a **direction** along the axis ("spin up" in physics lingo). □

Corollary 5.2. (of the reader's elegant proof in Corollary 5.1). The stabilizer subgroups $H_{\vec{u}} \subset \mathrm{SU}(2)$ of the points of the equator are the "longitudinal circles:"

$$a \pm (\sqrt{1 - a^2})\vec{u}$$

of S^3 since rotation around the direction $\vec{u}$ precisely stabilizes $\vec{u}$.

Exercises.

1. Check that the mapping:

$$z \mapsto \begin{bmatrix} (z+z^{-1})/2 & (z^{-1}-z)/2i \\ (z-z^{-1})/2i & (z+z^{-1})/2 \end{bmatrix}$$

is an isomorphism between $\mathbb{C}^*$ and $\mathrm{SO}(\mathbb{C},2)$.

2. Check that the subgroup $\{\pm\mathrm{id}\} \subset O(3,\mathbb{R})$ is normal, and conclude that there is a surjective homomorphism:

$$\phi : O(3,\mathbb{R}) \to \mathrm{SO}(3,\mathbb{R})$$

with kernel ± 1. Comparing this with the similar result in Corollary 5.1, decide whether or not $\mathrm{SU}(2)$ and $O(3,\mathbb{R})$ are isomorphic groups.

3. Given any symmetric *invertible* $n \times n$ matrix B, we can form the "orthogonal group with respect to B," i.e. the subgroup $O(n,B)$ of $\mathrm{GL}(n)$ consisting of the matrices A that "centralize" B:

$$A^T B A = B$$

Show that these are precisely the matrices that preserve "b-length" $b(\vec{v},\vec{v})$, where $b(\vec{v},\vec{w}) = (\vec{v})^T B \vec{w}$. Over the complex numbers, these groups are all isomorphic to each other, and hence are all isomorphic to $O(n,\mathbb{C})$ (can you prove this?) but investigate the example of:

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

over the real numbers.

4. An $m \times m$ matrix C is *skew-symmetric* if $C^T = -C$. Given an invertible skew-symmetric $m \times m$ matrix C, we can form the "symplectic group with respect to C" in exactly the same way, as the subgroup $\mathrm{Sp}(m,C) \subset \mathrm{GL}(m)$ of matrices that centralize the skew-symmetric form $c(\vec{v},\vec{w})$ defined in the same way by the matrix C. The "standard" symplectic group $\mathrm{Sp}(2n,k)$ over k is the group defined by:

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

i.e. with two block "anti-diagonal" $n \times n$ matrices of -1s and 1s.

(a) Prove that a skew-symmetric $m \times m$ matrix is never invertible if m is odd. Hence the symplectic groups are only defined for $m = 2n$.

(b) The group $\mathrm{Sp}(2,k)$ is isomorphic to a familiar group. Which one?