**Course Notes for Math 780-1 (Geometric Invariant Theory)**

**3. The Numerical Criterion, Points in $\mathbf{P}^1$ and Descent.** Suppose that $G$ is linearly reductive, that we are given a linearized $G$ action on a projective variety $X$ with respect to $L$, and that $x \in X$ is a closed point. The philosophy behind the numerical criterion is the idea that one can detect whether or not $x$ is stable by considering the induced (linearized) actions of all the inclusions $\mathbf{G}_m \subset G$. This will yield a very explicit procedure for determining whether $x$ is stable or not. We will use this procedure in an example to show which collections of $d$ unordered points on $\mathbf{P}^1$ are stable, semistable and unstable for the obvious linearized action of SL(2), and afterwards we will use brute force to explicitly describe the quotient in the case $d = 4$.

Next, we will prove a descent lemma of Kempf, which in particular tells us which power of $L$ descends to the GIT quotient. This is not an entirely trivial matter, as one will see from the example of four points on $\mathbf{P}^1$.

**Definition:** A non-trivial map $\lambda : \mathbf{G}_m \to G$ is a *one-parameter subgroup* (abbreviated 1-PS) of $G$.

Let $\widetilde{x}$ be a lift of $x$ to a non-zero point in $\widetilde{X}$, as in §2. If the morphism $\sigma_{\widetilde{x}} \circ \lambda : \mathbf{G}_m \to \widetilde{X}$ extends to a morphism $\alpha : \mathbf{A}^1 \to \widetilde{X}$ then we write $\lim_{t\to 0} \lambda(t)\widetilde{x} = \alpha(0)$. Otherwise, we will write $\lim_{t\to 0} \lambda(t)\widetilde{x} = \infty$. The following definition is clearly independent of the choice of $\widetilde{x}$.

**Definition:** Given a 1-PS $\lambda$, the point $x$ is:

(i) $\lambda$-*stable* if $\lim_{t\to 0} \lambda(t)\widetilde{x} = \infty$.

(ii) $\lambda$-*semistable* if $\lim_{t\to 0} \lambda(t)\widetilde{x} \neq 0$. (i.e. the limit is nonzero or $\infty$.)

(iii) $\lambda$-*unstable* if $\lim_{t\to 0} \lambda(t)\widetilde{x} = 0$.

**Note:** Since $X$ is proper, the map $\sigma_x \circ \lambda : \mathbf{G}_m \to X$ always extends to a map from $\mathbf{A}^1$ to $X$. We are saying here that $x$ is $\lambda$-stable iff that extension does not lift to $\widetilde{X}$.

**Theorem 3A (Hilbert's Numerical Criterion):** Assume $G =$ SL($n$) (but see the note at the end of the proof!)

(a) $x \in X^{SS}(L)$ if and only if $x$ is $\lambda$-semistable for *all* 1-PS $\lambda$.

(b) $x \in X^{S}(L)$ if and only if $x$ is $\lambda$-stable for *all* 1-PS $\lambda$.

**Proof:** In both parts one direction is relatively easy. Namely, if $x$ is not $\lambda$-semistable for some $\lambda$, then 0 is in the closure of the orbit of $\widetilde{x}$ under the action of $\mathbf{G}_m$, hence also under the action of $G$. So by definition, $x \notin X^{SS}(L)$.

Similarly, if $x$ is not $\lambda$-stable for some $\lambda$, then the map $\sigma_{\widetilde{x}} \circ \lambda : \mathbf{G}_m \to \widetilde{X}$ extends to a map from $\mathbf{A}^1$. On the other hand, every representation of $\mathbf{G}_m$ splits into one-dimensional invariant subspaces. So there is a basis for $\mathbf{C}^n$ under which the image of $\mathbf{G}_m$ has the form:

$$\operatorname{diag}\{t^{r_1}, ..., t^{r_n}\} := \begin{pmatrix} t^{r_1} & & & & 0 \\ & t^{r_2} & & & \\ & & \ddots & & \\ & & & t^{r_{n-1}} & \\ 0 & & & & t^{r_n} \end{pmatrix}$$

with $\sum r_i = 0$. Since some $r_i \neq 0$, the map $\lambda : \mathbf{G}_m \to G$ clearly cannot be extended to $\mathbf{A}^1$, and so $\sigma_{\widetilde{x}} : G \to \widetilde{X}$ cannot be proper, and $x \notin X^S(L)$.

To get the converses, we need to use the valuative criterion. That is, if we let $\mathcal{O} = \mathbf{C}[[t]]$ be the ring of formal-power series and $K = \mathbf{C}((t))$ be the field of fractions of $\mathcal{O}$, then the valuative criterion tells us:

(a) $\sigma_{\widetilde{x}} : G \to \widetilde{X}$ is not proper iff there is a morphism $\alpha :\ \operatorname{Spec}(K) \to G$ such that $\sigma_{\widetilde{x}} \circ \alpha$ extends to a morphism $\overline{\sigma_{\widetilde{x}} \circ \alpha} :\ \operatorname{Spec}(\mathcal{O}) \to \widetilde{X}$, but $\alpha$ does not extend across $\operatorname{Spec}(\mathcal{O})$.

(b) 0 is in the closure of $\sigma_x(G)$ iff there is a morphism $\alpha$ as above, where the closed point of $\operatorname{Spec}(\mathcal{O})$ is sent to 0 under $\overline{\sigma_{\widetilde{x}} \circ \alpha}$.

So suppose $x$ is not stable. Let $\alpha$ be the map whose existence is guaranteed by (a). Such a map is equivalent to an element of $\operatorname{SL}(n, K)$, that is, to a matrix $M(t)$ whose entries are rational power series in $t$. By the theory of elementary divisors (i.e. by row and column operations!) there are matrices $A(t)$ and $B(t)$ in $\operatorname{SL}(n, \mathcal{O})$ such that:

$$A(t)M(t)B(t) = \operatorname{diag}\{t^{r_1}, ..., t^{r_n}\}$$

where necessarily $\sum_{i=1}^n r_i = 0$ and because $\alpha$ did not extend, *some* $r_i$ *is nonzero*. Moreover, since our choice of basis of $\mathbf{C}^n$ was arbitrary, we may fix it by assuming that $B(0)$ is the identity.

Now, we define the 1-PS $\lambda$ by the requirement that $\lambda(t) = \text{diag}\{t^{r_1}, ..., t^{r_n}\}$ with respect to this basis for $\mathbf{C}^n$. That is, $A(t)M(t)B(t) = \lambda(t)$. We claim that $\lim_{t\to 0} \lambda(t)\widetilde{x} \neq \infty$.

The action of $G$ on the affine cone $\widetilde{X} \subset \mathbf{A}^{N+1}$ is induced by a *linear* action of $G$ on $V := \Gamma(X, L)^*$ by definition, so the action of our 1-PS $\lambda$ on $V$ diagonalizes. Let $e_0, ..., e_N$ be a basis for $V$ with the property that $\lambda(t)e_i = t^{s_i}e_i$ with $s_0 \leq ... \leq s_N$. Since $B(0)$ is the identity, if we let $\hat{b}_{i,j}(t)$ be the (rational power series) entries of the induced action of $B^{-1}(t)$ on $V$ with respect to this basis, then in particular, $\hat{b}_{i,j}(0) = \delta_{i,j}$.

Then if we write $\widetilde{x} = \sum x_i e_i$, we have:

$$\begin{array}{rclcl} A(t)M(t)\widetilde{x} & = & \sum_i x_i A(t)M(t)e_i & = & \sum_i x_i \lambda(t)B^{-1}(t)e_i \\ & = & \sum_i \left(\sum_j \hat{b}_{i,j}(t)x_j\right)\lambda(t)e_i & = & \sum_i \left(\sum_j \hat{b}_{i,j}(t)x_j\right)t^{s_i}e_i \end{array}$$

By assumption, this has a finite limit as $t \to 0$, but since $\hat{b}_{i,j}(0) = \delta_{i,j}$, this implies that $x_i = 0$ if $s_i < 0$, which is to say that $\widetilde{x}$ is not $\lambda$-stable(!)

If in addition $\lim_{t\to 0} M(t)\widetilde{x} = 0$, then in the previous paragraph, $x_i = 0$ if $s_i \leq 0$, and $\widetilde{x}$ is not $\lambda$-semistable. This completes the proof of Theorem 3A.

**Remark:** In the proof, the theory of elementary divisors was used to show that given $M \in \text{SL}(n, K)$, there are elements $A, B \in \text{SL}(n, \mathcal{O})$ such that $AMB$ is a 1-PS. This was the only place in the proof where we used the fact that $G = \text{SL}(n, \mathbf{C})$, as opposed to any linearly reductive group. Iwahori's Theorem says that this is also true for any linearly reductive $G$, so the numerical criterion always applies.

The following example classically went under the catchy title of "binary quantics".

**Points on $\mathbf{P}^1$:** Consider the symmetric action $\sigma : \text{SL}(2) \times \mathbf{P}^d \to \mathbf{P}^d$ coming from the isomorphism $\text{Sym}^d(\mathbf{P}^1) \cong \mathbf{P}^d$. If we let $V \cong \mathbf{C}^2$, then $\sigma$ this is naturally linearized by the action $\widetilde{\sigma} : \text{SL}(V) \times \text{Sym}^d(V^*) \to \text{Sym}^d(V^*)$. The numerical criterion immediately gives the stable and semi-stable points of this action.

Suppose $\lambda$ is a 1-PS, and $X_0, X_1$ is a basis for $V^*$ with respect to which $\lambda(t) = \text{diag}\{t^{-r}, t^r\}$. Then via $\lambda$, $\mathbf{G}_m$ acts on the monmials $M_{d-i,i} := X_0^{d-i}X_1^i$ by $M_{d-i,i} \mapsto t^{r(2i-d)}$.

Thus the polynomial $P(X_0, X_1) = \sum a_i X_0^{d-i} X_1^i \neq 0$ lies over a $\lambda$-semistable point of $\mathbf{P}^d$ if and only if $a_i \neq 0$ for some $i \leq \frac{d}{2}$, that is, if and only if $X_1$ is a root of order at most $\frac{d}{2}$. The polynomial $P$ lies over a $\lambda$-stable point of $\mathbf{P}^d$ if and only if $X_1$ is a root of order strictly less than $\frac{d}{2}$.

Therefore by the numerical criterion, $P(X_0, X_1)$ lies over a semistable point of $\mathbf{P}^d$ if and only if every root of $P$ has multiplicity at most $\frac{d}{2}$. $P(X_0, X_1)$ lies over a stable point if and only if every root of $P$ has multiplicity strictly less than $\frac{d}{2}$.

Suppose that $d$ is even, so $(\mathbf{P}^d)^S \neq (\mathbf{P}^d)^{SS}$. Then I claim that there is a unique closed orbit in $(\mathbf{P}^d)^{SS} - (\mathbf{P}^d)^S$. Indeed, if $X_1$ is a root of order exactly $\frac{d}{2}$ of $P(X_0, X_1)$, then via the 1-PS $\lambda$ defined above, we see that $a_{\frac{d}{2}} X_0^{\frac{d}{2}} X_1^{\frac{d}{2}}$ is in the closure of the orbit of $P(X_0, X_1)$. Since $\mathrm{SL}(V)$ acts transitively on pairs of points, it is easy to see that the orbit corresponding to a pairs of points of multiplicity $\frac{d}{2}$ is the unique closed orbit which is semistable but not stable.

**Four Points on $\mathbf{P}^1$:** You are certainly already familiar with the GIT quotient of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ by $\mathrm{SL}(2)$ with respect to $\mathcal{O}(1,1,1,1)$ (!)

Given four *ordered* points $z_1, z_2, z_3, z_4 \in \mathbf{C}$, recall that the cross-ratio $\frac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ yields a rational map $\mathbf{C}^4 --> \mathbf{C}$ which is *constant on the orbits of the action of PGL(1)*. We can projectivize the cross-ratio to get a rational map:

$$\phi : (\mathbf{P}^1)^4 --> \mathbf{P}^1$$

$$((z_1, w_1), ..., (z_4, w_4)) \mapsto ((z_4w_1 - z_1w_4)(z_3w_2 - z_2w_3), (z_4w_2 - z_2w_4)(z_3w_1 - z_1w_3))$$

which is constant on the orbits of the action of $\mathrm{PGL}(1)$, and defined precisely on the locus where no three of the points of $\mathbf{P}^1$ coincide.

On the other hand, the same analysis as above shows that the semi-stable points of the action of $\mathrm{SL}(2)$ on $(\mathbf{P}^1)^4$ are precisely those where no three points coincide. It is then immediate that $\phi$ is the quotient coming from GIT.

Notice that if we let $F = (z_4w_1 - z_1w_4)(z_3w_2 - z_2w_3)$ and if we let $G = (z_4w_2 - z_2w_4)(z_3w_1 - z_1w_3)$, then both $F$ and $G$ are invariant under $\mathrm{SL}(2)$, and the target $\mathbf{P}^1$ is canonically isomorphic to $\mathrm{Proj}[F, G]$, and therefore, finally, $\phi^*\mathcal{O}(1) = \mathcal{O}(1,1,1,1)$.

Now, let's turn to four *unordered* points.

If $\psi : \mathbf{P}^4 -\!-\!> C$ is the GIT quotient, then from the example of four ordered points, it follows that $\mathbf{P}^1$ maps onto $C$, so $C$ is rational. In order to find the map $\psi$, we are looking for homogeneous polynomials in $(a_0, ..., a_4)$ which are invariant under the action of SL(2) on $P(X_0, X_1)$. Unlike the previous example, one can check that there are no linear invariants. In fact:

**Claim:** If $\mathcal{O}_{\mathbf{P}^4}(d) = \psi^*(A)$, then $6|d$.

**Proof:** Consider the point $x \in \mathbf{P}^4$ determined by $P(X_0, X_1) = (X_0 - X_1)(X_0 - \omega X_1)(X_0 - \omega^2 X_1)X_1$ where $\omega$ is a cube root of unity. Then the element $\mathrm{diag}\{\omega, \omega^2\} \in$ SL(2) fixes $x$ but acts by multiplication by $\omega^2$ on $P(X_0, X_1)$.

Now if $\mathcal{O}_{\mathbf{P}^4}(d) = \psi^*(A)$, then because it is a pullback from the quotient, SL(2) must act trivially on $\mathcal{O}_{\mathbf{P}^4}(d)$. Thus, this example shows that $3|d$. Similarly, one shows that $2|d$ by considering the appropriate element of SL(2) acting on $P(X_0, X_1) = (X_0 - X_1)(X_0 - iX_1)(X_0 + X_1)(X_0 + iX_1)$. (You should convince yourself that this does *not* give $4|d$, as you might naively expect!)

On the other hand, the wonderful nineteenth century mathematicians discovered (and this is really what the study of binary quantics is all about) that the two homogeneous polynomials:

$F = \frac{1}{6}(a_2^2 - 3a_1a_3 + 12a_0a_4)$ and
$G = a_0a_2a_4 - \frac{3}{8}a_0a_3^2 - \frac{3}{8}a_1^2a_4 + \frac{1}{8}a_1a_2a_3 - \frac{1}{36}a_2^3$

generate the full ring of invariants. One should check that (as must be the case!) $F$ and $G$ determine the locus where there is a zero of multiplicity three, and that the discriminant $D$, which is, after all, another invariant, is expressible as $D = \alpha(P^3 - 6Q^2)$, where $\alpha \in \mathbf{C}$. (See Mumford's article in "Algebraic Geometry, Oslo 1970" if you are lazy.)

But now, we see that $\psi$ maps to $\mathbf{P}^1 = \mathrm{Proj}[F, G] = \mathrm{Proj}[F^3, G^2]$ by $\psi(a_0, ..., a_4) = (F^3(a_0, ..., a_4), G^2(a_0, ..., a_4))$, so $C = \mathbf{P}^1$, and $\mathcal{O}_{\mathbf{P}^4}(6) = \psi^*(\mathcal{O}_{\mathbf{P}^1}(1))$. (Notice that the unique non-stable point is where the discriminant vanishes, namely the point $(6, 1) \in \mathbf{P}^1$.)

In light of this last example, it seems worthwhile to find a general procedure for determining which twist of $L$ descends to the GIT quotient under a linearized $G$-action. This will follow from Kempf's descent lemma, which (surprisingly!) seems to only recently have been proved in the form given here by Drezet and Narasimhan:

**Definition:** A vector bundle $F$ of rank $r$ on $X$ is a $G$-bundle if a linear lift to $F$ of the action of $G$ on $X$ is understood. A map $\widetilde{\sigma} : G \times F \to F$ is a linear lift of $\sigma$ if, for all $x \in X, t \in F_x$, and $A \in \mathrm{GL}(r)$, $\widetilde{\sigma}(g,t) \in F_{\sigma(g,x)}$, and $\widetilde{\sigma}(g, At) = A\widetilde{\sigma}(g,t)$.

**Example:** The linearization of a $G$-action on $X$ is equivalent to regarding $L$ as a $G$-bundle. Of course the latter has the advantage of making sense even when $L$ is not very ample(!) In fact, GIT and the numerical criterion apply to linearizations of $G$-actions for any ample (even big and nef?) line bundle. One finds that if $L = L_0^{\otimes d}$, and $G$ is linearlized with respect to $L$ by the $d$th tensor of the linearization with respect to $L_0$, then the GIT quotients are the same.

**Definition:** A $G$-bundle $F$ descends to the GIT quotient of $X$ under a linearized $G$-action if the restriction of $F$ to $X^{SS}(L)$ is isomorphic as $G$-bundle to the pull-back of a vector bundle on the quotient.

Suppose $X$ is a projective variety acted upon by $G$, a linearly reductive group, suppose the action is linearized with respect to an ample line bundle $L$, and suppose that $F$ is a $G$-bundle. Let $\phi : X^{SS}(L) \to Y$ be the GIT quotient associated to the inclusion $R^G \subset R$, as in §2. Then:

**(Kempf's Descent Lemma) Theorem 3B:** $F$ descends to a vector bundle on the GIT quotient of $X$ if and only if the stabilizer of $x$ acts trivially on $F_x$ for every $x \in X^{SS}(L)$ such that $O(x)$ is closed.

**Proof:** If $F$ descends, then it is clear that $G_x$ acts trivially on $F_x$. To prove the converse, it suffices to find, for each $y \in Y$, an affine neighborhood $V_0$ of $y$, and $r$ $G$-invariant sections $t_1, ..., t_r$ spanning $F|_{\phi^{-1}(V_0)}$.

Fix $y \in Y$ a closed point.

**Step 1:** If $x \in \phi^{-1}(y)$ is a closed point such that $O(x)$ is closed, then there are $r$ $G$-invariant sections of $F|_{O(x)}$ which span $F_{gx}$ for every $g \in G$. Indeed, since we assumed that $G_x$ acts trivially on $F_x$, we can translate a basis for $F_x$ by $G$ to obtain the desired sections.

**Step 2:** Let $V = D(f)$ for $f \in \mathrm{Proj}(R^G)$ be an affine neighborhood of $y$. Then $\pi^{-1}(V) = D(f) \subset X$ is also affine. We claim that there is a Reynolds operator $E : \mathrm{H}^0(\phi^{-1}(V), F) \to \mathrm{H}^0(\phi^{-1}(V), F)^G$. To see this, by Lemma 1.1 it suffices to show that $G$ acts rationally on $\mathrm{H}^0(\phi^{-1}(V), F))$. But if we choose an affine, dense $U \subset \pi^{-1}(V)$ on which $F$ trivializes, then an $s \in \mathrm{H}^0(U, F)$

gives rise to a regular function $h : G \times U \to \mathbf{C}^r$ defined by $(g, x) \mapsto gs(g^{-1}x)$. Then if $G = \operatorname{Spec}(A)$ and $U = \operatorname{Spec}(B)$, we have $h = \sum a_i \otimes \bar{b}_i$, where $a_i \in A$ and $\bar{b}_i : U \to \mathbf{C}^r$, and $Gs|_U$ is contained in the span,$W$, of the $\bar{b}_i$. Since the restriction of sections from $\phi^{-1}(V)$ to $U$ is injective, we prove rationality by intersecting $W$ with $\mathrm{H}^0(\phi^{-1}(V), F)$.

**Step 3:** Take the sections $s_1, ..., s_r$ spanning $F|_{O(x)}$ from Step 1 and extend them to sections $\bar{s}_1, ..., \bar{s}_r$ of $F|_{\phi^{-1}(V)}$, which is possible since this is affine. Apply the Reynolds operator of Step 2 to get invariant sections $t_1, ..., t_r$, which by Lemma 1.1 still restrict to $s_1, ..., s_r$ on $O(x)$. Finally, consider the closed invariant subset $Z \subset \phi^{-1}(V)$ where $t_1, ..., t_r$ fail to span $F$. The image $\phi(Z) \subset V$ is closed and proper, since it does not contain $\phi(x)$, so we can shrink $V$ to an affine $V_0$ for which $t_1, ..., t_r$ have the desired properties.