**Course Notes for Math 780-1 (Geometric Invariant Theory)**

## 2. Quotients, Orbits and GIT.

### The Affine Case

Suppose that a group $G$ acts on an affine variety $X$. We want to know when a "good" affine quotient exists. In light of the previous section, we should take $G$ to be (linearly) reductive, but even more, if we want the quotient to correspond to our intuitive notion of a quotient, for example, with each fiber corresponding to a single closed orbit, then we may have to restrict the action of $G$ to an open subset of $X$. We begin with Mumford's definition of the best possible quotient.

**Definition:** A pair $Y$ and $\phi : X \to Y$ is a *geometric quotient* of the action of $G$ on a quasiprojective variety $X$ if $Y$ is quasiprojective, $\phi$ is a morphism, and:

(i) $\phi$ is surjective, and for each closed point $y \in Y$, the fiber $\phi^{-1}(y)$ is a single closed orbit.

(ii) For each invariant open subset $U \subset X$, there is an open subset $V \subset Y$ such that $U = \phi^{-1}(V)$.

(iii) For each open subset $V \subset Y$, $\phi^* : \Gamma(V, \mathcal{O}_Y) \to \Gamma(\phi^{-1}(V), \mathcal{O}_X)$ is an isomorphism onto the subring of invariants $\Gamma(\phi^{-1}(V), \mathcal{O}_X)^G$.

Consider the action of $\mathrm{SL}(W)$ on $\mathbf{A}^{kr} \cong \mathrm{Hom}(V, W)$ of the introduction. Since 0 is in the closure of the orbit of every non-surjective map, it is immediate that no geometric quotient can exist. (After all, fibers of a morphism are closed!) On the other hand, the orbits corresponding to surjective homomorphisms are closed with trivial stabilizer, and the union of all such is an open subvariety of $\mathbf{A}^{kr}$. This is a model for the following definition:

**Definition:** If $x \in X$ is a closed point, then the orbit, $O(x)$, of the action of $G$ on $X$ is *stable* if:

(i) $O(x)$ is closed, and

(ii) the stabilizer, $G_x$, of $x$, is finite.

This doesn't quite correspond to Mumford's notion of stability in GIT, but it is easier to work with in part because of the following useful equivalent condition for stability.

**Lemma 2.1:** If $G$ is a linear group acting on an affine variety $X$, then the orbit of a closed point $x$ is stable if and only if the induced map $\sigma_x : G \to X$ is proper.

**Proof:** If $\sigma_x$ is proper, then of course $O(x)$ is closed, and the stabilizer $G_x = \sigma_x^{-1}(x)$ is complete and affine ($G$ is affine!), hence finite.

On the other hand, if $O(x)$ is closed, then to prove properness of $\sigma_x$, it suffices to show that $\sigma_x : G \to O(x)$ is a finite morphism. Since the fibers are finite, this is easy to check generically, that is, for the restriction of $\sigma_x$ to $\sigma_x^{-1}(U)$ for some non-empty open subset $U \subset O(x)$ (is this an exercise in Hartshorne?). Translation of $U$ by elements of $G$ then shows that $\sigma_x$ is finite everywhere.

Now, suppose that $G$ is a linearly reductive group acting on an affine variety $X$. Let $R = \Gamma(X, \mathcal{O}_X)$, and since $G$ acts rationally on $R$, the ring of invariants $R^G$ is finitely generated, by Theorem 1A. Let $Y = \operatorname{Spec}(R^G)$, and let $\phi : X \to Y$ be the dominant morphism induced by the inclusion of rings: $R^G \subset R$. Then we have the:

**(Affine GIT) Theorem 2A:** The morphism $\phi$ has the following properties:

(a) $\phi$ is surjective.

(b) If $x$ and $x'$ are closed points of $X$, then $\phi(x) = \phi(x')$ if and only if the closure of their orbits has nonempty intersection.

(c) For each closed point $y \in Y$, the fiber $\phi^{-1}(y)$ contains a unique closed orbit (but potentially many orbits that are not closed!).

(d) There is an invariant open subset $X^S \subset X$ such that $x \in X^S$ if and only if $O(x)$ is a stable orbit. Then $Y^S := \phi(X^S)$ is open in $Y$, and the pair $Y^S$, together with $\phi|_{X^S} : X^S \to Y^S$ is a geometric quotient of the action of $G$ on $X^S$.

**Proof:** We start with a geometric version of Corollary 1.4. Namely, suppose that $(Z_i)$ is a family of closed, invariant subsets of $X$. Then by Corollary 1.4,

$$\overline{\phi(\cap_i Z_i)} = \cap_i \overline{\phi(Z_i)}.$$

Suppose that $Z \subset X$ is closed and invariant, and $y \in \overline{\phi(Z)}$ is a closed point. Then $\phi^{-1}(y)$ is also closed and invariant, so

$$\overline{\phi(Z \cap \phi^{-1}(y))} = \overline{\phi(Z)} \cap \{y\} = \{y\}$$

and $Z \cap \phi^{-1}(y) \neq \emptyset$ implies that $y \in \phi(Z)$, so $\phi(Z)$ is closed.

Parts (a) and (b) of the theorem now follow immediately. The map $\phi : X \to Y$ is dominant, so since $\phi(X)$ is closed, it must be all of $Y$, and we have (a). If $\phi(x) = \phi(x') = y$, then $\phi(\overline{O(x)} \cap \overline{O(x')}) = \phi(\overline{O(x)}) \cap \phi(\overline{O(x')}) = \{y\}$, so $\overline{O(x)} \cap \overline{O(x')} \neq \emptyset$. This gives (b).

Suppose $O(x)$ is an orbit of minimal dimension. The complement $Z := \overline{O(x)} - O(x)$ is invariant, of smaller dimension, so if $x' \in Z$, then its orbit would have smaller dimension, contradicting minimality. So $O(x)$ is closed. Uniqueness follows immediately from (b).

Consider the morphism $\Psi = (\sigma, \mathrm{id}) : G \times X \to X \times X$ where $\sigma$ is the group action. If $(x, x)$ is a closed point in the diagonal, the fiber $\Psi^{-1}(x, x)$ is isomorphic to the stabilizer $G_x$. Moreover, there is a section of $\Psi$ over the diagonal given by $(x, x) \mapsto (1, x)$. Thus we can apply uppersemicontinuity at the section, and because the fibers are groups, hence equidimensional, we have:

$$X^{reg} := \{x \in X | G_x \text{ is of minimal dimension}\}$$

is invariant and open in $X$.

If the minimal dimension of $G_x$ is positive, there is nothing to prove. If there exist points with finite stabilizers, then $Y^S = Y - \phi(X - X^{reg})$ and $X^S = \phi^{-1}(Y^S)$, so $Y^S$ is open in $Y$, and $X^S$ is open in $X$.

Finally, we need to prove that the map $\phi : X^S \to Y^S$ is a geometric quotient. Property (i) is immediate from (a). For property (ii), suppose $U \subset X^S$ is open and invariant. Then $Z = X - U$ is closed and invariant, so $\phi(Z)$ is closed in $Y$. Let $V = Y^S - \phi(Z)$. Then $\phi|_{X^S}^{-1}(V) \subset U$, but if $x \in U$, then $O(x)$ is closed and invariant, in $X^S$, and $Z \cap O(x) = \emptyset$, so $\phi(x) \in V$. This gives (ii). Finally, suppose that $V = D(f)$ is the open affine subset of $Y$ defined by the nonvanishing of $f \in R^G$. Then $\Gamma(V, \mathcal{O}_Y) = R_f^G$, and $\Gamma(\phi^{-1}(V), \mathcal{O}_X) = R_f$. But it is a simple consequence of the Reynolds identity (Corollary 1.2) that $R_f^G$ is the ring of invariants of $R_f$. Property (iii) follows.

## The Projective Case

Suppose that $X$ is a projective variety, and that $L$ is a very ample line bundle on $X$. Suppose that $G$ acts on $X$ and fixes the bundle $L$, and that the action of $G$ is lifted to the total space of $L$. By this I mean that there is an action of $G$ on the affine cone $\widetilde{X}$ over the embedding of $X$ by the complete linear series, which descends to the action on $X$, and such that the lift commutes with multiplication by scalars. If we let $R$ denote the (graded) image of $\sum \Gamma(X, L)^{\otimes d}$ in $\sum \Gamma(X, L^{\otimes d})$ under the multiplication, then there is an induced action of $G$ on $R$.

We will use the lifted action of $G$ (which is called a *linearization* of the action of $G$ in the literature) to produce a quotient of $X$ by the action of $G$. By restricting to stable orbits (to be defined!), we may produce a geometric quotient, as in the affine case. However, such a quotient will often not be projective. If we insist that our quotient be a projective scheme (which is of obvious interest in our applications to moduli questions!), we will have to be satisfied with a weaker notion of a quotient.

**Definition:** If $G$ acts on a quasiprojective variety $X$, then a *categorical quotient* of $X$ by the action of $G$ is a pair consisting of a quasiprojective $Y$ and a surjective morphism $\phi : X \to Y$ satisfying:

(i) $\phi$ is constant on the orbits of the closed points of $X$.

(ii) Given a variety $T$ and a morphism $\psi : X \to T$ which satisfies (i), then there is a unique morphism $\kappa : Y \to T$ such that $\psi = \kappa \circ \phi$.

It is immediate that a categorical quotient is unique. The following criterion is useful for detecting categorical quotients:

**Lemma 2.2:** If $G$ acts on $X$, and $\phi : X \to Y$ is a morphism which is constant on orbits, then it is a categorical quotient if:

(i) For all open $V \subset Y$, $\phi^*(\Gamma(V, \mathcal{O}_Y)) = \Gamma(\phi^{-1}(V), \mathcal{O}_X)^G$, and

(ii) If $Z \subset X$ is invariant and closed, then $\phi(Z)$ is closed. If $(Z_i)_{i \in I}$ is a set of invariant closed subsets of $X$, then

$$\phi(\bigcap_{i \in I} Z_i) = \bigcap_{i \in I} \phi(Z_i)$$

**Proof:** (i) implies $\phi$ is dominant, so (ii) implies it is surjective. Suppose that $\psi : X \to T$ is constant on orbits. We need to construct $\kappa : Y \to T$. Choose an affine open cover $\{W_i\}$ of $T$, and let $Z_i = X - \psi^{-1}(W_i)$. These are closed and invariant in $X$, so by (ii), $V_i = Y - \phi(Z_i)$ are open, and $\phi^{-1}(V_i) \subset \psi^{-1}(W_i)$. Since $\{W_i\}$ is an open cover of $T$, it follows that $\cap Z_i = \emptyset$, so by (ii), $\cap\phi(Z_i) = \emptyset$, so $\{V_i\}$ is an open cover of $Y$.

If $\kappa : Y \to T$ satisfies $\psi = \kappa \circ \phi$, then we must have $\kappa(V_i) \subset W_i$ for all $i$, so the restriction of $\kappa$ to each $V_i$ is determined by a ring homomorphism $f_i : \Gamma(W_i, \mathcal{O}_T) \to \Gamma(V_i, \mathcal{O}_Y)$. But by property (ii), $\Gamma(V_i, \mathcal{O}_Y)$ injects as the invariant subring of $\Gamma(\phi^{-1}(V_i), \mathcal{O}_X)$, which contains the invariant subring of $\Gamma(\psi^{-1}(W_i), \mathcal{O}_X)$, so the $f_i$ are *uniquely determined* by the requirement that $\psi = \kappa \circ \phi$. Thus the resulting maps from $V_i$ to $W_i$ are uniquely defined, and glue together to give $\kappa$.

**Corollary 2.3:** A geometric quotient is a categorical quotient. Hence in particular, geometric quotients are unique.

**Proof:** A geometric quotient satisfies the conditions of the lemma.

**Corollary 2.4:** The morphism $\phi : X \to Y$ of affine varieties in Theorem 2A is a categorical quotient.

**Proof:** In the proof of Theorem 2A, we verified all the conditions of the lemma for $\phi : X \to Y$.

Let us return to the action of $G$ on $X$ linearized with respect to the very ample line bundle $L$. If $x \in X$ is a closed point, then we let $\widetilde{x} \in \widetilde{X}$ denote a nonzero lift of $x$ to the affine cone. The following definitions are easily seen to be independent of the choice of lift:

**Definition:** (i) $x \in X$ is unstable (with respect to the linearization) if $0 \in \overline{O(\widetilde{x})}$. Let $X^U(L) \subset X$ be the set of unstable points.

(ii) $x \in X$ is semistable if $x \notin X^U(L)$. Let $X^{SS}(L) = X - X^U(L)$.

(iii) $x \in X$ is stable if $O(\widetilde{x})$ is closed with finite stabilizer. If $x$ is stable, it is obviously semistable. Let $X^S(L) \subset X^{SS}(L) \subset X$ be the set of stable points.

If $G$ is linearly reductive in the above discussion, then the ring of invariants $R^G$ is a finitely generated (Theorem 1A again!) sub-graded ring of $R$, so we can construct the projective scheme $Y = \text{Proj}(R^G)$, and *rational* map $\phi : X --> Y$ coming from the inclusion $R^G \subset R$. Then:

**(Projective GIT) Theorem 2P:** The map $\phi$ has the following properties:

(i) $\phi$ is defined on $X^{SS}(L)$, and is surjective.

(ii) $\phi : X^{SS}(L) \to Y$ is a categorical quotient.

(iii) The image $Y^S := \phi(X^S(L))$ is open in $Y$, and $\phi : X^S(L) \to Y^S$ is a geometric quotient.

(iv) For $x$ and $x'$ in $X^{SS}(L)$, $\phi(x) = \phi(x')$ if and only if $\overline{O(x)} \cap \overline{O(x')} \cap X^{SS}(L) \neq \emptyset$.

(v) If $Z \subset X^{SS}(L)$ is closed and invariant, then $\phi(Z) \subset Y$ is closed.

**Proof:** Consider the closed invariant subset $Z \subset X$ defined by the vanishing of the invariant sections in the image of the multiplication $\Gamma(X, L)^{\otimes d} \to \Gamma(X, L^{\otimes d})$. If we consider the action of $G$ on the affine cone $\widetilde{X}$, then it follows from part (b) of Theorem 2A that $Z = X^U(L)$. So we have (a).

The affine open sets $D(f)$ for invariant $f$ in $R$ are an open cover of $X^{SS}(L)$. But each such open set is invariant, and moreover the restriction of $\phi$ to $D(f)$ is precisely the affine GIT quotient of Theorem 2A associated to the inclusion $(R^G)_f \subset R_f$ (Reynolds identity again!). As with Corollary 2.4, part (ii) now follows from Lemma 2.2 and the proof of Theorem 2A, since the conditions of the lemma are all checkable locally on $Y$.

Similarly part (iii) follows from the local statement on $Y$, provided we can show that if $x \in X^S(L)$, then for every $f$ not vanishing at $x$, $x \in D(f)$ has a closed orbit with finite stabilizer. But if the orbit of $\widetilde{x}$ is closed in $\widetilde{X}$, then for any other $y \in D(f)$ and nonzero lift $\widetilde{y}$, there is an element of $R^G$ vanishing on $\widetilde{x}$, but not on $\widetilde{y}$, by the affine GIT. But this element localizes to separate $x$ and $y$ in $D(f)$, so $O(x)$ is closed. The stabilizer of $O(x)$ cannot be infinite, since $O(\widetilde{x}) \to O(x)$ has finite fibers, so $x \in D(f)$ is stable.

Finally, (iv) and (v) follow from the local versions on $Y$, which have already been proved.

Suppose $C$ is a smooth curve, $p \in C$ is a closed point, and $f : C - p \to X^{SS}(L)$. Then the completion of the map to $\overline{f} : C \to X$ may send $p$ to $X^U(L)$. However, we have the following corollary of Theorem 2P:

**Corollary (Semi-Stable Replacement Property):** There is a (possibly not projective!) curve $C'$, maps $\pi : C' \to C$, $h : C' - \pi^{-1}(p) \to G$, and a point $q \in \pi^{-1}(p)$ such that the completion of the map:

$$f' : C' - \pi^{-1}(p) \to X^{SS}(L) \text{ defined by } f'(x) = h(x)f(\pi(x))$$

sends $q$ to $X^{SS}(L)$.

**Proof:** Since $\phi(X^{SS}(L))$ is projective, we can always complete the map $\phi \circ f : C \to Y$ to get the *image* of a semi-stable point as the image of $p$. We want to factor the completed map $\overline{\phi \circ f}$ through $X^{SS}(L)$. Consider the image $V := \sigma(G \times C - p)$ in $X^{SS}(L)$. The closure of $V$ is closed and invariant in $X^{SS}(L)$, therefore its image in $Y$ is closed, so it contains $\phi(f(p))$. Let $x \in \overline{V}$ be a point lying over $\phi(f(p))$. There must be a curve $S$ and $q \in S$ such that $S - q \subset G \times C - p$ and $\lim_{s \to q} \sigma(s) = x$ (Hartshorne?). But now the projection of $S - q$ to $C - p$ is nonconstant (!) and indeed completes to take $q$ to $p$. So we are done.

**Note:** In the corollary, we can always arrange it so that $f'(q)$ is in the unique closed orbit over $\phi(f(p))$.