Moduli in Algebraic Geometry: An Introduction

Math 7800, Spring 2022. Instructor: Aaron Bertram

0.3. Coherent Sheaves. On an affine Noetherian scheme Spec(A), each finitely generated A-module M defines a coherent sheaf $\widetilde{M}$ with

$$\widetilde{M}(U_f) = M_f$$

on the basis of open subsets $U_f, f \in A$ of Spec(A). The stalk of this sheaf at a prime ideal P is the localized module M_P (as a module over the stalk $\mathcal{O}_{X,P} = A_P$.)

A coherent sheaf $\mathcal{F}$ on a general Noetherian scheme X is a sheaf of $\mathcal{O}_X$-modules that restricts to sheaves $\widetilde{M}_i$ on an open affine cover $U_i = \text{Spec}(A_i)$ and a morphism of coherent sheaves is a morphism of sheaves of $\mathcal{O}_X$-modules. On an affine scheme, a morphism $f : M \to N$ of A-modules uniquely determines a morphism $\widetilde{\alpha} : \widetilde{M} \to \widetilde{N}$ of coherent sheaves and vice versa, i.e. the "tilde" operation is an equivalence of categories between finitely generated A-modules and coherent sheaves on Spec(A).

The category of coherent sheaves on X is abelian, equipped with a *tensor product* (of sheaves of $\mathcal{O}_X$-modules). When $X = \text{Proj}(R_\bullet)$, each finitely generated graded $R_\bullet$-module $M_\bullet$ similarly determines a coherent sheaf $\widetilde{M}$ via graded localizations $\widetilde{M}(U_F) = M_F$, the degree zero part of the localization of M at the multiplicative set generated by F. In this case, two graded modules determine isomorphic coherent sheaves if and only if they agree in all sufficiently high degrees. Thus in this case the passage from graded modules to coherent sheaves induces an equivalence relation on finitely generated graded modules.

Examples. The twisted module $S(d)_\bullet$ on $\mathbb{P}^n_k$ defined by:

$$S(d)_e = k[x_0, ..., x_n]_{d+e}$$

corresponds to the coherent sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$, which is *invertible* via

$$\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(-d) \cong \mathcal{O}_{\mathbb{P}^n}$$

We have already encountered the coherent *ideal sheaves* $\mathcal{I}_Z \subset \mathcal{O}_X$ for closed subschemes $Z \subset X$ of a Noetherian scheme.

To form the cokernel $\mathcal{C}$ of a morphism $\alpha : \mathcal{F} \to \mathcal{G}$ of coherent sheaves on X, it is not sufficient to take cokernels $\mathcal{C}(U) = \mathcal{F}(U)/\mathcal{G}(U)$ of the sections. Instead, this object is only a pre-sheaf which is sheafified (reduced to stalks and rebuilt) in order to define the cokernel sheaf. The tensor product similarly requires sheafification, but it is actually more problematic than this. The "correct" tensor product of two coherent sheaves is the derived tensor product, which produces an object of the derived category of X (see §0.4).

A coherent sheaf $\mathcal{F}$ on a Noetherian scheme X is:

(a) **locally free** if X has a (finite) cover by open sets U such that:

$$\mathcal{F}|_U \cong \oplus^r \mathcal{O}_X|_U$$

are free modules over the rings of regular functions. These are the vector bundles.

(b) **invertible** if $\mathcal{F}_U \cong \mathcal{O}_X|_U$ above. These are the line bundles.

Remark. On an affine scheme S*pec*(A), it is the **projective** modules M over A (and not just the free modules) that give rise to locally free coherent sheaves $\widetilde{M}$.

The **fiber** of a coherent sheaf $\mathcal{F}$ at a point $x \in X$ is the vector space:

$$\mathcal{F}(x) := \mathcal{F}_x \otimes_{k(x)} \mathcal{O}_{X,x}$$

where $k(x)$ is the residue field $\mathcal{O}_{X,x}/m_x$ (the quotient by the maximal ideal). It is a consequence of Nakayama's Lemma that if $v_1, ..., v_r \in \mathcal{F}(x)$ are a spanning set of vectors, then the induced map of coherent sheaves:

$$\mathcal{O}_X|_U^r \to \mathcal{F}|_U$$

defined in a neighborhood $x \in U$ is surjective, and then, working with the fibers of the kernel coherent sheaf, that in a (smaller) neighborhood there is a presentation:

$$\mathcal{O}_X|_V^s \stackrel{\Phi}{\to} \mathcal{O}_X|_V^r \to \mathcal{F}|_V$$

with $\Phi = (\phi_{ij})$ a matrix of regular functions. The corank of $\Phi(x)$ is the rank of $\mathcal{F}(x)$ since the tensor product is right exact. It follows that:

$$x \mapsto \dim \mathcal{F}(x)$$

is upper semi-continuous (taking its minimum on an open set), but even more, the minors of Φ define closed **subschemes** $Z_r \subset X$ and a stratification of X with locally closed strata $T_r = Z_r - Z_{r+1}$ that are maximal with the property that:

$$\mathcal{F} \text{ is locally free of rank } r \text{ on } T_r$$

Given a morphism $f : X \to Y$ of Noetherian schemes, the **pushforward**

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

of a coherent sheaf on X defines a *quasi-coherent* sheaf $f_*\mathcal{F}$ of $\mathcal{O}_Y$ modules, which shares all the properties of a coherent sheaf except that the modules M_i need not be finitely generated. When f is a projective morphism, however, the push-forward is coherent, though again it is not really the correct object when there are *higher direct image* coherent sheaves. As a functor, the push-forward is left exact.

In the affine case, when $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine schemes, and $\widetilde{N}$ is a coherent sheaf on X, then $f_*\widetilde{N} = \widetilde{N}$, regarding the B-module N as an A-module via the ring homomorphism $\alpha : A \to B$.

The **pull-back** $f^*\mathcal{G}$ of a coherent sheaf on Y is defined locally by:

$$f^*\mathcal{G} = f^{-1}(\mathcal{G}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

where $f^{-1}(G)(U) = \lim_{V \supseteq f(U)} \mathcal{G}(V)$ defines a pre-sheaf of $f^{-1}(\mathcal{O}_Y)$-modules, which becomes a coherent sheaf of $\mathcal{O}_X$-modules via the tensor product. As a functor, the pull-back is right exact.

In the affine case as above, $f^*\widetilde{M} = \widetilde{M \otimes_A B}$, converting the A-module M to a B-module (with the same generators).

Notice that locally free sheaves pull back to locally free sheaves, and the pull-back is **exact** on sequences of locally free sheaves, but it is generally not left exact.

Example. The exact sequence of coherent sheaves on X:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

for a closed embedding $i : Z \hookrightarrow X$ pulls back to:

$$i^*\mathcal{I}_Z \to \mathcal{O}_Z \stackrel{\sim}{\to} \mathcal{O}_Z \to 0$$

and the pull-back of the ideal sheaf is not the zero sheaf.

In fact, if $f : X \to S$ is a separated S-scheme, then:

$$\Omega_{X/S} = \Delta^*(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)$$

is the coherent sheaf of relative differentials for $\Delta : X \to X \times_S X$. In general, the coherent sheaf $i^*(\mathcal{I}_Z)$ is the conormal sheaf of the closed embedding.

Example. If E is a locally-free sheaf of rank r on S, then $\pi : \mathbb{P}(E) \to S$ is the *bundle of projective spaces* equipped with a surjective map $\pi^*E \to \mathcal{O}_{\mathbb{P}(E)}(1)$ onto a line bundle representing the functor:

- $\mathbb{P}(E)(T) = \{a^*E \to \mathcal{L} \to 0\}$ for S-schemes $a : T \to S$ and
- $\mathbb{P}(E)(u) =$ pull-back for morphisms $u : U \to T$ of S-schemes.

then the relative dualizing sheaf for π fits into the Euler sequence:

$$0 \to \Omega_{\mathbb{P}(E)/S} \to \pi^*E \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathcal{O}_{\mathbb{P}(E)} \to 0$$

It is a locally free sheaf by Nakayama's Lemma. One can similarly define the bundle of Grassmaannians (or flag varieties) over S. A *relative proj* construction is used to create the S-scheme $\mathbb{P}(E) \to S$, which is locally (over S) isomorphic to projective space $\mathbb{P}^r_U$ over (small enough) open sets U.

(i) If $\mathcal{L}$ is an invertible sheaf on S, then:

$$\mathbb{P}(E \otimes \mathcal{L}) = \mathbb{P}(E) \text{ and } \mathcal{O}_{\mathbb{P}(E\otimes\mathcal{L})}(1) = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{L}$$

(ii) A surjective map $F \to E$ of locally free sheaves defines a closed embedding:

$$i : \mathbb{P}(E) \subset \mathbb{P}(F) \text{ of } S\text{-schemes, with } i^*\mathcal{O}_{\mathbb{P}(F)}(1) = \mathcal{O}_{\mathbb{P}(E)}(1)$$

In particular, if E is generated by global sections, i.e. if there is a surjection:

$$\mathcal{O}_S^n \to E \text{ for some } n$$

then $\pi : \mathbb{P}(E) \subset \mathbb{P}^{n-1}_S \to S$ is a projective morphism.

Definition. An invertible sheaf $\mathcal{L}$ on S is **ample** if for each coherent sheaf $\mathcal{F}$, there is a $n_\mathcal{F} \in \mathbb{Z}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ are generated by global sections for all $n \ge n_\mathcal{F}$.

Thus if S admits an ample line bundle, then each $\pi : \mathbb{P}(E) \to S$ is projective.

One source of ample line bundles are the (very) ample line bundles coming from embeddings in projective space. Namely:

Theorem A (Serre). Let X be of finite type over k. Then an invertible sheaf $\mathcal{L}$ on X is ample if and only if $\mathcal{L}^{\otimes d} = i^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ for some embedding $i : X \hookrightarrow \mathbb{P}^r_k$.

The embedding need not be closed in the Theorem (e.g. S could be affine).

If X is proper, though, then the embedding must be closed, and in that case X is projective if and only if X has an ample line bundle. In general, a line bundle $\mathcal{L}$ on an S-scheme $f : X \to S$ is *relatively ample* if $\mathcal{L}^{\otimes d} = i^*\mathcal{O}_{\mathbb{P}^n_S}(1)$ for some closed embedding of S-schemes.

Suppose $X \subset \mathbb{P}^r_k$ is a projective scheme over k. Then $\mathcal{F} = \widetilde{M_\bullet}$ for:

$$M_d = \Gamma(X, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(d))$$

for the global section functor Γ from coherent sheaves to vector spaces. When $d \ge d_\mathcal{F}$, the Hilbert function $\dim(M_d)$ is a **polynomial** in d. This is the Hilbert polynomial $H_\mathcal{F}(d)$, which (as a reality check) only depends on $M_\bullet$ in large degrees.

We seek a relative version of the Hilbert polynomial for projective S-schemes. This is remarkably given to us by the notion of *flatness*.

Recall that a module M over a commutative ring A is flat if $M \otimes_A \bullet$ is exact:

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

(an exact sequence of A-modules) remains exact when tensored by M:

$$0 \to M \otimes N_1 \to M \otimes_A N_2 \to M \otimes_A N_3 \to 0$$

As with the pull-back, this is generally not the case, and in fact if M is finitely generated and A is local, then M is flat over A if and only if M is free. Thus a finitely generated module M over a Noetherian ring A is flat if and only if each M_P is flat over A_P (localizing is exact), if and only if M is locally free (i.e. projective). However, we are primarily interested in modules that are not finitely generated:

Remark. It is a useful exercise to convince yourself that M is flat if and only if:

$$M \otimes_A I \to M \text{ is injective for all ideals } I \subset A$$

Definition. (i) A coherent sheaf $\mathcal{F}$ on an S-scheme

$$f : X \to S$$

is **flat** over S at $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module (via $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$).

(ii) $\mathcal{F}$ is flat over S it is flat over S at all points $x \in X$.

(iii) If $\mathcal{O}_X$ itself is flat over S, then we say $f : X \to S$ is a flat morphism.

Remark. Flatness itself is stable under base extension. That is, if $\mathcal{F}$ is flat over S and $a : T \to S$ is a morphism, then:

$$\mathcal{F}_T := \widetilde{a}^* \mathcal{F}$$

is flat over T, for the fiber product diagram:

$$\begin{array}{ccc} X_T & \stackrel{\widetilde{a}}{\to} & X \\ \downarrow & & f \downarrow \\ T & \stackrel{a}{\to} & S \end{array}$$

Two examples of flatness are explicit and very useful.

Flatness over a Field. If $\mathcal{O}_{S,s}$ is a field, then flatness places no condition on a module $\mathcal{F}_x$ with $f(x) = s$, which, as a module over $\mathcal{O}_{S,s}$, is simply a vector space.

Flatness over a DVR. If $\mathcal{O}_{S,s}$ is a DVR, flatness requires that the generator $\pi \in m_s$ not annihilate any (nonzero) element of $\mathcal{F}_x$.

Remark. Each coherent sheaf $\mathcal{F}$ on a Noetherian scheme X has a finite set of *associated points*, which are the associated primes of the modules M_i on an affine cover of X. Then the two conditions above give the following:

Flatness over a Regular Curve. If C is a regular curve over a field k, then a coherent sheaf on $f : X \to C$ is is flat over C if and only if each associated point of $\mathcal{F}$ maps to the generic point of C.

Example. Suppose X is a variety of finite type over k. Then $f : X \to C$ is flat if and only if f is not the constant map. If X is reduced, then f is flat if and only if every component of X maps to the generic point.

A more difficult result characterizes flat *projective* morphisms.

Remark. The *fibers* of an S-morphism $f : X \to S$ are the subschemes:

$$i_s : X_s = X \times_S \operatorname{Spec}(k(s)) \hookrightarrow X$$

for the (closed) points of S, and the fibers of a coherent sheaf $\mathcal{F}$ are: $\mathcal{F}|_{X_s} := i_s^*\mathcal{F}$.

Then there is a natural map of vector spaces: $f_*\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} k(s) \to \Gamma(X_s, \mathcal{F}|_{X_s})$ from the fiber of the push-forward sheaf to the global sections of the fiber sheaf. In general, this may be neither injective nor surjective. However, when f is flat, this is an **isomorphism** (of vector spaces).

Flatness for Projective Morphisms. If S is of finite type over a field k and

$$\pi : X \hookrightarrow \mathbb{P}^n_S \to S$$

is a projective S-morphism, then $\mathcal{F}$ is flat over S if and only if the coherent sheaves:

$$\pi_* \left(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_S}(d)\right)$$

are locally free for all $d \geq d_{\mathcal{F}}$. In particular, by the previous remark, the *Hilbert polynomials* of the fiber sheaves $\mathcal{F}_s$ over a connected base are **constant**.

Corollary 1. The dimension of the fibers of a flat morphism $f : X \to S$ is constant.

Corollary 2. Each coherent sheaf $\mathcal{F}$ on $f : X \to S$ as above induces a canonical *flattening stratification* on S, consisting of "maximal" locally closed subschemes:

$$T_P \text{ indexed by polynomials } P : \mathbb{Z} \to \mathbb{Z}$$

over which $\mathcal{F}$ is flat of relative Hilbert polynomial P.

The Hilbert Scheme Functor for $\mathbb{P}^n_k$.

- $h_P(S) \subset \{\text{closed embeddings } X \subset \mathbb{P}^n_S\}$

consists of flat S-schemes $f : X \to S$ of relative Hilbert polynomial P.

- $h(a) =$ base extension by $a : T \to S$

This was generalized by Grothendieck to

The Quotient Functor for a coherent sheaf $\mathcal{V}$ on a closed subscheme $Z \subset \mathbb{P}^n_k$

- $Q_P(S) = \{\text{quotient sheaves } \mathcal{V}_S \to \mathcal{F} \text{ on } Z \times S \text{ that are flat over } S\}$
- $h(a) =$ base extension.

We will prove these are representable by projective schemes of finite type over k.