Abstract Algebra. Math 6320. Bertram/Utah 2022-23.
Irreducible Complex Representations

We start with:

Schur's Lemma. If V_1 and V_2 are irreducible complex representations of G, then:

(a) $\mathrm{Hom}_G(V_1, V_2) = 0$ or else

(b) $\mathrm{Hom}_G(V_1, V_2) = \mathbb{C} \cdot f$ is spanned by a (G-linear) isomorphism $f : V_1 \to V_2$.

Proof. Suppose $f \in \mathrm{Hom}_G(V_1, V_2)$. Because $\ker(f)$ is invariant, it follows that $\ker(f) = 0$ or else $\ker(f) = V_1$ i.e. f is either injective or the zero map. Similarly, the invariant image $f(V_1) \subset V_2$ shows that either $f(V_1) = 0$ or $f(V_1) = V_2$, so f is either the zero map or f is surjective. Thus f either 0 or an isomorphism.

Now suppose $g \in \mathrm{Hom}_G(V_1, V_2)$ is another G-linear isomorphism, and consider the *automorphism* $\sigma = g^{-1} \circ f : V_1 \to V_1$. Now consider the one-parameter family:

$$f_\lambda := \sigma - \lambda \cdot \mathrm{id}_{V_1} \in \ \mathrm{Hom}_G(V_1, V_1) \text{ for } \lambda \in \mathbb{C}$$

Then by the same reasoning as in the first paragraph, each of these G-linear maps is either an automorphism or else the zero map. But if λ is an eigenvalue of σ (which always exists, since $\mathbb{C}$ is algebraically closed), then f_λ is not an isomorphism, so it must be zero. Thus:

$$g \circ f^{-1} = \sigma = \lambda \cdot \mathrm{id}_V \text{ for some } \lambda$$

and then $g = \lambda \cdot f$, as desired. $\square$

Corollary. Suppose V is a complex representation of G and:

$$V = U_1 \oplus \cdots \oplus U_r$$

is a direct sum of irreducible representations $U_1, ..., U_r$ (possibly with repetitions). Then $\mathrm{Hom}_G(V, U) \neq 0$ for an irreducible U if and only if $U \cong U_i$ for some i.

Proof. If U is not isomorphic to any of the U_i and $f \in \mathrm{Hom}_G(V, U)$, then each of the restricted maps: $f|_{U_i} : U_i \to U$ is the zero map by Schur's lemma, and so:

$$f(v) = f(u_1 + \cdots + u_r) = f(u_1) + \cdots + f(u_r) = 0$$

for all $v = u_1 + \cdots + u_r \in V$. In other words, f itself is the zero map.

On the other hand, if $U = U_i$ for some i, then the *projection map*

$$f(v) = f(u_1 + \cdots + u_i + \cdots + u_r) = u_i$$

is well-defined and G-invariant, and evidently not the zero map. $\square$

Given a G-representation U, let $nU = U \oplus U \oplus \cdots \oplus U$ (n times).

Corollary. Given a finite complex representation V of a finite group G, let:

$$V = n_1 U_1 \oplus \cdots \oplus n_l U_l$$

collecting isomorphic irreducibles. If $U \cong U_i$, then $\dim_k(\mathrm{Hom}_G(V, U)) = n_i$.

Proof. This uses the second part of Schur's lemma. Let $f : U_i \to U$ be an isomorphism (uniquely determined up to the choice of scalar). Then a G-linear map $h : V \to U$ is completely determined by scalars $(\lambda_1, ..., \lambda_{n_i})$ via:

$$h|_{n_i U_i}(u_1 + \cdots + u_{n_i}) = \sum_{j=1}^{n_i} \lambda_j f(u_j) \text{ and } h|_{n_j U_j} = 0 \text{ for } j \neq i \quad \square$$

Corollary. If G is finite (so direct sum decompositions exist), then:

$$V \cong \bigoplus_U \dim_k(\mathrm{Hom}_G(V, U))U$$

summed over all irreducible representations U. That is, the decomposition into irreducibles, with multiplicities, is determined by the morphism spaces.

We now apply this to a particular representation.

The Regular Representation. Let G be a finite group and define:

$$\mathbb{C}[G] = \langle e_g \mid g \in G \rangle$$

to be the complex vector space $\mathbb{C}^{|G|}$ with one basis vector for each element $g \in G$. This vector space comes equipped with the "regular" representation of G given by:

$$\rho(h)(e_g) = e_{hg}$$

i.e. $\rho(h)$ permutes the basis vectors by left translation by the element $h \in G$.

Example. We've seen the regular representation of C_n in the previous section.

We can think of $\mathbb{C}[G]$ as the vector space of *functions* $f : G \to \mathbb{C}$, in which e_g are the "delta functions" $e_g(h) = 0$ if $h \neq g$ and $e_g(g) = 1$ and an arbitrary function $f : G \to \mathbb{C}$ is then a sum:

$$f = \sum_{g \in G} f(g) e_g \in \mathbb{C}[G]$$

From this point of view, the regular representation is the action of the group G on the vector space of functions taking:

$$h \cdot f = f \circ \text{left translation by } h^{-1}$$

which is exactly the content of: $h \cdot e_g = e_{hg}$. (Note the inverse!)

Thinking of elements of $\mathbb{C}[G]$ as functions may or may not help when thinking about the regular representation of finite groups, but it gives us a road map for what to do with groups (e.g. orthogonal or unitary groups) that are not finite. The idea is to limit the space of all functions $f : G \to \mathbb{C}$ (which is too big) to more manageable G-invariant subspaces. These will not be finite-dimensional, since in fact orthogonal and unitary groups have infinitely many irreducible complex representations, but this idea helps to find the countably many irreducible complex representations.

Theorem. The regular representation $\mathbb{C}[G]$ of G satisfies:

$$\mathrm{Hom}_G(\mathbb{C}[G], U) \cong U \text{ for all irreducible } U \text{, so that } \mathbb{C}[G] = \sum_U \dim(U) \cdot U$$

summed over **all** irreducible complex representations U.

Proof. Let $u \in U$, and define a G-linear map $h_u : \mathbb{C}[G] \to U$ by setting:

$$h_u(e_g) = gu \text{ for each basis vector } e_g$$

where $gu \in U$ is the result of acting u by the group element g. In other words:

$$h_u(\sum_{g \in G} \lambda_g e_g) = \sum_{g \in G} \lambda_g gu \text{ for each vector } v = \sum_{g \in G} \lambda_g e_g \in \mathbb{C}[G]$$

Then for all $g' \in G$ and all basis vectors e_g of $\mathbb{C}[G]$, we have:

$$h_u(g'e_g) = h_u(e_{g'g}) = (g'g)u = g'(gu) = g'h_u(g)$$

so h_u is G-linear. Moreover, $h_u(e_{\mathrm{id}}) = \mathrm{id} \cdot u = u$ for $\mathrm{id} \in G$, so if we define:

$$\mathrm{ev}: \ \mathrm{Hom}_G(\mathbb{C}[G], U) \to U \text{ by } \mathrm{ev}(h) = h(e_{\mathrm{id}}) =: u_h$$

then $h = h_{u_h}$ and h is an isomorphism, with inverse $u \mapsto h_u$. $\square$

The Theorem has two powerful consequences:

Corollary.

(1) Every irreducible representation of G is a summand of $\mathbb{C}[G]$. Even more:

(2) The dimensions of the irreducible representations U of G satisfy:

$$\dim(\mathbb{C}[G]) = |G| = \sum_U \dim(U)^2$$

Examples. (a) The only irreducible representations of an abelian group G are characters, so: $|G| = 1^2 + 1^2 + \cdots + 1^2$ and indeed G has $|G|$ complex characters.

(b) We have found three irreducible representations of $S_3 = D_6$:

- The trivial representation (dimension one)
- The sign character (also dimension one)
- The symmetries of the equilateral triangle (dimension two)

Since $6 = 1^2 + 1^2 + 2^2$, there are no others!

(c) We have found two irreducible representations of D_8:

- The trivial representation (dimension one)
- The symmetries of the square (dimension two)

Since $8 = 1^2 + 2^2 + 3$ and 3 is only a sum of squares in one way ($3 = 1^2 + 1^2 + 1^2$), there are three other characters. Let's find them. The subgroup

$$H = \{1, x^2\} \subset D_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

is **normal** since $x(x^2)x^{-1} = x^2$ and $y(x^2)y^{-1} = x^{-2}yy^{-1} = x^2$, and the quotient is the group of left cosets:

$$D_8/H = \{H, xH.yH, xyH\}$$

Since $x^2 \in H$, $y^2 = 1$ and $(xy)^2 = 1$, each of the cosets $\{xH, yH, xyH\}$ has order two in D_8/H, and so D_8/H is the Klein group $C_2 \times C_2$. This is an abelian group, so it has four characters (see (a)). Each of them determines a character of D_8 via:

$$D_8 \to D_8/H \to \mathbb{C}^*$$

These are the four characters (including the trivial character) of D_8.

(d) In addition to the irreducible three-dimensional representation of A_4 as the symmetries of the tetrahedron, we can use the normal Klein subgroup:

$$K_4 = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \lhd A_4$$

to find three characters of A_4 via the map $A_3 \to A_4/K_4 = C_3$, and $12 = 9+1+1+1$, so we know these are all of them.

The notion of a *generalized character* of a representation will help us to be more systematic about finding the irreducible representations of a finite group. This is the **trace** of a complex representation ρ:

$$\chi_\rho : G \to \mathbb{C} \text{ given by } \chi_\rho(g) = \mathrm{tr}(\rho(g))$$

i.e. $\chi_\rho(g)$ is the trace of the matrix associated to $\rho(g)$ (which does not depend upon the choice of basis). If $\chi : G \to \mathbb{C}^*$ is itself a (one-dimensional) character, then the trace is χ itself.

Note that the trace applied to the identity element of G is always:

$$\chi_\rho(\mathrm{id}) = \dim(V)$$

the dimension of the representation.

Example. (a) The character of the two-dim'l irreducible representation ρ of S_3 is:

$$\begin{array}{lcccc}
\chi_\rho(\mathrm{id}) & = & \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & = & 2 \\[2ex]
\chi_\rho(1\ 2) & = & \mathrm{tr}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} & = & 0 \\[2ex]
\chi_\rho(2\ 3) & = & \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} & = & 0 \\[2ex]
\chi_\rho(1\ 3) & = & \mathrm{tr}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} & = & 0 \\[2ex]
\chi_\rho(1\ 2\ 3) & = & \mathrm{tr}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & = & -1 \\[2ex]
\chi_\rho(1\ 3\ 2) & = & \mathrm{tr}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} & = & -1
\end{array}$$

and we see in particular that a generalized character can take the value 0. In fact:

(b) The character of the regular representation $\mathbb{C}[G]$ is:

$$\chi_\rho(\mathrm{id}) = |G| \text{ and } \chi_\rho(g) = 0 \text{ for } \textbf{all } g \neq \mathrm{id}$$

To see the this, note that:

$$\rho(g)(e_h) = e_{gh} \neq e_h \text{ if } g \neq \mathrm{id}$$

and so the matrix for the action of g is a permutation matrix with no fixed basis vectors, i.e. there are only zeroes on the diagonal, so the trace is zero.

Definition. A function $\alpha : G \to \mathbb{C}$ is a *class function* if $\alpha(h) = \alpha(ghg^{-1})$ for all $g, h \in G$, i.e. if α is constant on the conjugacy classes of G.

Examples. (a) Let $C_1, ..., C_r \subset G$ be the conjugacy classes of G. The functions:

$$\delta_i(g) = \begin{cases} 1 \text{ if } g \in C_i \\ 0 \text{ otherwise} \end{cases}$$

are step functions that form a basis for the vector space of class functions.

(b) Each character $\chi_\rho : G \to \mathbb{C}$ of a representation ρ is a class function since

$$tr(\rho(g)\rho(h)\rho(g)^{-1}) = tr(\rho(h)) \text{ for all } g \text{ and } h$$

Theorem. Define an inner product on the space $Z[G]$ of class functions by:

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\beta(g)$$

Then the following *orthogonality relations* hold among the characters:

(i) The characters of irreducible representations of G are orthonormal and

(ii) The characters of irreps are an orthonormal **basis** of $Z[G]$.

Example. Before we prove this, let's see the Theorem in practice for $G = S_3$.

- The conjugacy classes of S_3 are:

$$C_1 = \{\text{id}\}, \text{C}_2 = \{(1\ 2), (2\ 3), (1\ 3)\}, \text{C}_3 = \{(1\ 2\ 3), (1\ 3\ 2)\}$$

- The irreducible representations of S_3 are:

χ_{tr} (trivial), χ_{sgn}(sign), and the two-dimensional irreducible representation ρ

We arrange the characters of these representations in a **character table**.

	id	(1 2)	(1 2 3)
	1	3	2
χ_{tr}	1	1	1
χ_{sgn}	1	-1	1
ρ	2	0	-1

- The first row is a list of representatives from each conjugacy class C_i
- The second row is a list of the sizes $|C_i|$ of each conjugacy class.
- The first column is a list of the irreducible representations
- The rest of the table computes the characters of the representations.

You may now check the orthogonality relations. For example:

$$(\chi_\rho, \chi_\rho) = \frac{1}{6}\left(2^2 + 2(-1)^2\right) = 1 \text{ and } (\chi_{tr}, \chi_{sgn}) = \frac{1}{6}\left(1 + 3 \cdot (-1) + 2 \cdot 1\right) = 0$$

The key idea in the proof is to notice that $\text{Hom}(V, W)$ (not G-linear!) is itself a G-representation whenever V and W are G-representations. The action of G is:

$$(gf)(v) = g(f(g^{-1}v))$$

so that if $f(v) = w$, then we have $(gf)(gv) = gw$. This is a G-representation since $((hg)f)((hg)v) = (hg)w = h(gw) = h((gf)(gv)) = (h(gf))((hg)v)$ and in particular f is G-linear if and only if $(gf) = f$ for **all** $g \in G$, i.e.

- $\text{Hom}_G(V, W)$ is the largest subspace of $\text{Hom}(V, W)$ on which G acts trivially

The character of $\text{Hom}(V, W)$ is the product:

$$\chi_{\text{Hom(V,W)}}(g) = \chi_V(g^{-1}) \cdot \chi_W(g)$$

as one can check choosing bases of eigenvalues for $g^{-1} : V \to V$ and $g : W \to W$.

In addition, notice that the character of the representation $V \oplus W$ satisfies:

$$\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$$

so that, for example, $\chi_{\mathbb{C}[G]}(g) = \sum_U \dim(U) \cdot \chi_U(g)$.

Proof. This relies on two averaging maps. Given a representation (V, ρ), define

$$p : V \to V \text{ by setting } p(v) = \frac{1}{|G|}\sum_{g\in G} gv. \text{ Then}$$

(a) $hp(v) = p(v)$, so the image of p consists of G-invariant vectors of V

(b) $p(v) = v$ for all G-invariant vectors in V.

Thus $p : V \to V_G \subset V$ is the *projection* onto the subspace of G-invariant vectors, and it follows that $tr(p) = \dim(V_G)$. On the other hand,

$$tr(p) = \frac{1}{|G|}\sum_{g\in G} tr(\rho(g)) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)$$

is the average of the values of the character of V. When we apply this to the space $\mathrm{Hom}(V, W)$ of maps between representations,

$$\dim(\mathrm{Hom}_G(V, W)) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g^{-1})\chi_W(g)$$

If V and W are **irreducible** representations, then from Schur's Lemma we get:

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g^{-1})\chi_W(g) = \left\{ \begin{array}{l} 1 \text{ if } V = W \\ 0 \text{ if } V \not\cong W \end{array} \right.$$

which is very nearly (i) of the Theorem. In fact, it is (i) since the characters of representations of a finite group satisfy $\chi_V(g^{-1}) = \overline{\chi_V(g)}$. To see this, recall that each matrix $A = \rho(g)$ has finite order, hence all its eigenvalues are roots of unity, and roots of unity satisfy $\zeta^{-1} = \overline{\zeta}$. But then the trace of A is a sum $\sum \zeta_i$ of roots of unity, and the trace of A^{-1} is $\sum \zeta_i^{-1} = \sum \overline{\zeta}_i = \overline{tr(A)}$.

We prove (ii) with a weighted average function. For a class function $\alpha \in Z[G]$,

$$p_\alpha : V \to V \text{ is defined by } p_\alpha(v) = \frac{1}{|G|}\sum_{g\in G} \alpha(g) g \cdot v.$$

Then we claim that $p_\alpha \in \mathrm{hom}_G(V, V)$. Indeed,

$$p_\alpha(h\cdot v) = \frac{1}{|G|}\sum_{g\in G} \alpha(g)(gh)\cdot v = \frac{h}{|G|}\sum_{g\in G} \alpha(g)(h^{-1}gh)\cdot v = \frac{h}{|G|}\sum_{g\in G} \alpha(h^{-1}gh)(h^{-1}gh)\cdot v$$

since α is a class function, and then this is just a reordering of the sum, so:

$$\frac{h}{|G|}\sum_{g\in G} \alpha(h^{-1}gh)(h^{-1}gh) \cdot v = h \cdot p_\alpha(v)$$

If V is an irreducible representation, then $p_\alpha = \lambda \cdot \mathrm{id}_V$ by Schur's Lemma, and:

$$\lambda \cdot \dim(V) = tr(p_\alpha) = \frac{1}{|G|}\sum_{g} \alpha(g)\chi_V(g) = (\overline{\alpha}, \chi_V)$$

so that in particular, if $(\overline{\alpha}, \chi_V) = 0$, then $\lambda = 0$ and $p_\alpha = 0$.

If the characters of irreducible representations **fail** to be a basis for $Z[G]$, then there is a nonzero class function $\alpha \in Z[G]$ such that $\overline{\alpha}$ is *orthogonal* to all characters. That is $(\overline{\alpha}, \chi_V) = 0$ for all irreducible representations. But then by linearity, $(\overline{\alpha}, \chi_V) = 0$ for **all** characters, and $p_\alpha = 0$ for all representations of G. If we apply this to the regular representation $\mathbb{C}[G]$, we get:

$$p_\alpha(e_{\mathrm{id}}) = \frac{1}{|G|}\sum_g \alpha(g)e_g = 0$$

from which we conclude that $\alpha(g) = 0$ for all values of g. This is a contradiction. $\square$

After doing all this work, let's have some fun.

Character table for A_4. The conjugacy classes of A_4 are:

$$C_1 = \{\mathrm{id}\}, C_2 = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

$$C_3 = \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\}, C_4 = \{(1\ 3\ 2), (1\ 4\ 3), (1\ 2\ 4), (2\ 4\ 3)\}$$

and we've seen that the irreducible representations of A_4 are $\chi_{tr}, \chi_\omega, \chi_{\omega^2}, \rho$ where $\omega = e^{2\pi i/3}$ and $\chi_\omega(1\ 2\ 3) = \omega$ and $\chi_{\omega^2}(1\ 2\ 3) = \omega^2$ and $\rho(g)$ is a symmetry of the tetrahedron in three-space. Then the character table of A_4 is:

	id	(1 2)(3 4)	(1 2 3)	(1 3 2)
	1	3	4	4
χ_{tr}	1	1	1	1
χ_ω	1	1	ω	ω^2
χ_{ω^2}	1	1	ω^2	ω
ρ	3	-1	0	0

where the traces of $\rho(g)$ are computed as follows:

(i) $\rho((1\ 2)(3\ 4))$ is a rotation by π about an axis, with eigenvalues $1, -1, -1$.

(ii) $\rho(1\ 2\ 3)$ is a rotation by either $2\pi/3$ or $4\pi/3$ about an axis, with eigenvalues $1, \omega, \omega^2$ and trace zero. and similarly for $\rho(1\ 3\ 2)$.

When we have an incomplete list of irreducible representations, there are various methods for filling in the table. If we are missing one representation, then its character (but not the representation itself) can be deduced from the orthogonality relations. Other methods for finding new representations include:

Multiplying by a One-dimesional Character. If (V, ρ) is a representation of G and χ is a one-dimensional character of G, then:

$$\rho'(g) = \chi(g) \cdot \rho(g) : V \to V$$

is another representation. If V is irreducible, then $\chi \cdot \rho$ is irreducible, and:

$$\chi'_\rho = \chi \cdot \chi_\rho$$

because multiplying a matrix by a scalar multiplies the trace by the same scalar. Then by the orthogonality relations, we know that:

$$(V, \rho') \cong (V, \rho) \text{ if and only if } \chi_{\rho'} = \chi_\rho$$

Thus, for example, the square of the sign character is always the trivial character, and if $\rho : S_3 \to \mathrm{Aut}(\mathbb{C}^2)$ is the irreducible two-dimensional representation, then (reading from the character table)

$$\chi(\rho) = (2, 0, -1) \text{ and } \chi_{sgn} = (1, -1, 1), \text{ so } \chi_{\rho'} = (2, 0, -1)$$

and multiplying by the sign does not produce a new representation.

Character Table for S_4. The five conjugacy classes of S_4 have representatives:

$$\text{id}, (1\ 2), (1\ 2\ 3), (1\ 2\ 3\ 4), (1\ 2)(3\ 4)$$

and via the quotient group $S_4/K_4 = S_3$ and the symmetries of the cube ρ_{cub}, we count the three irreducible representations of S_3 plus ρ_{cub} among the irreducible representations of S_4. This gives the following partial character table:

	id	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
	1	6	8	6	3
χ_{tr}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
ρ_{tri}	2	0	-1	0	2
ρ_{cub}	3	-1	0	1	-1
ρ	3				

The fact that the missing character ρ is three-dimensional follows from:

$$24 = 1^2 + 1^2 + 2^2 + 3^2 + \dim(\rho)^2$$

and from the orthogonality relations, we obtain the full last line of the table:

$$(3, 1, 0, -1, -1)$$

and this is indeed the character of the missing representation $\chi_{sgn} \cdot \rho_{cub}$.

Another interesting way of obtaining new representations is by:

Automorphisms of G. If $\sigma : G \to G$ is a symmetry (in the category of groups), and $\rho : G \to \text{Aut}(V)$ is a representation, then the **composition**:

$$\rho \circ \sigma : G \to G \to \text{Aut}(V)$$

is a representation. We've seen in §5 that **conjugating** by $g \in G$ is a symmetry:

$$\sigma_g(h) = ghg^{-1}$$

but these "inner" automorphisms of G do not change conjugacy classes, and thus do not change characters of representations. However, groups do on occasion have "outer" automorphisms that do change characters of representations.

Character Table for A_5. The five conjugacy classes of A_5 have representatives:

$$\text{id},\ (1\ 2)(3\ 4),\ (1\ 2\ 3), (1\ 2\ 3\ 4\ 5), (1\ 3\ 5\ 2\ 4)$$

and the outer automorphism σ obtained by conjugating A_5 by the odd permutation (2 3 5 4) exchanges the last two conjugacy classes (while fixing the others). Consider the representation ρ_{dod} of A_5 given by the action of A_5 on the dodecahedron. Then:

$\rho_{dod}((1\ 2)(3\ 4))$ is a rotation by π, with trace $1 + (-1) + (-1) = -1$.

$\rho_{dod}(1\ 2\ 3)$ is a rotation by $2\pi/3$ or $4\pi/3$, with trace $1 + \omega + \omega^2 = 0$.

$\rho_{dod}(1\ 2\ 3\ 4\ 5)$ is a rotation by $2m\pi/5$, since it is an element of order 5. If it is by $2\pi/5$ or $8\pi/5$, then the trace is $1+\tau+\tau^4 = \phi = (1+\sqrt{5})/2$, the golden mean, where $\tau = e^{2\pi i/5}$. If it is by $4\pi/5$ or $6\pi/5$, then the trace is $1+\tau^2+\tau^3 = (1-\sqrt{5})/2 = 1-\phi$. Whichever rotation is taken by $\rho_{dod}(1\ 2\ 3\ 4\ 5)$, the square $(1\ 3\ 5\ 2\ 4)$ is taken to a rotation with the opposite trace. So the character of ρ_{dod} is either:

$$(3, -1, 0, \phi, 1-\phi) \text{ or } (3, -1, 0, 1-\phi, \phi)$$

and indeed, composing with the symmetry σ takes one to the other.

Thus, we have three out of five rows of the character table for A_5:

	id	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
	1	15	20	12	12
χ_{tr}	1	1	1	1	1
ρ_{dod}	3	-1	0	ϕ	$1-\phi$
$\rho_{dod} \circ \sigma$	3	-1	0	$1-\phi$	ϕ
ρ_1					
ρ_2					

and once again, the dimension count will tell us the other two dimensions:

$$60 = 1^2 + 3^2 + 3^2 + \dim(\rho_1)^2 + \dim(\rho_2)^2$$

from which it follows that $\dim(\rho_1) = 4$ and $\dim(\rho_2) = 5$ since:

$$41 = 16 + 25 \text{ is the unique way to express 41 as a sum of squares}$$

So we seek four and five dimensional irreducible representations of the group A_5. For the first, consider the permutation representation:

$$\rho_{perm}(\sigma)(e_i) = e_{\sigma(i)} \text{ for } V = \langle e_1, e_2, ..., e_5\rangle$$

The trace of $\rho_{perm}(\sigma)$ is the number of elements fixed by σ, so $\chi_{\rho_{perm}} = (5, 1, 2, 0, 0)$. Is this the missing five-dimensional irreducible representation? No!

$$(\chi_{\rho_{perm}}, \chi_{\rho_{perm}}) = \frac{1}{60}\left(5^2 + 15(1^2) + 20(2^2)\right) = 2$$

But we knew this wasn't irreducible anyway, since:

$$e_1 + e_2 + e_3 + e_4 \text{ is an invariant vector for the permutation action}$$

from which it follows that the representation (V, ρ_{perm}) satisfies

$$V = U \oplus W \text{ where } U \text{ is the one-dimensional trivial representation}$$

This gives us W with $\chi_W = \chi_V - \chi_U = (4, 0, 1, -1, -1)$ and **this** is irreducible. With the orthogonality relations, we can finish off the table:

	id	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 3 5 2 4)
	1	15	20	12	12
χ_{tr}	1	1	1	1	1
ρ_{dod}	3	-1	0	ϕ	$1-\phi$
$\rho_{dod} \circ \sigma$	3	-1	0	$1-\phi$	ϕ
$\rho_{st} = \rho_{perm}/\chi_{tr}$	4	0	1	-1	-1
μ	5	1	-1	0	0

with the character of the missing representation μ.

Remark. The permutation representations of S_n and A_n always split off a trivial summand. The remaining representation is the **standard representation** ρ_{st}.

One way to find the missing representation μ is to consider the representation:

$$\hom(V_{dod}, V_{dod\circ\sigma})$$

which has character: $\chi_{dod} \cdot \chi_{dod\circ\sigma} = (9, 1, 0, -1, -1)$ and $(\chi_\mu, \chi_\mu) = 2$ so χ_μ has dimension 9 and is a sum of two irreducibles. Thus $\hom(V_{dod}, V_{dod\circ\sigma}) = V_{st} \oplus V_\mu$. This method of finding new representations is made precise with tensor products.