7. Riemann-Roch. Let $D$ be a divisor (not necessarily effective) on a non-singular curve $C \subset \mathbb{C}P^n$. Recall that:

$$L(D) = \{ \phi \in K(C)^* \mid \text{div}(\phi) + D \geq 0 \} \cup \{0\} \subset K(C)$$

is a finite-dimensional vector space over $\mathbb{C}$ of dimension $l(D)$.

**Theorem 7.1 (“Classical” Riemann-Roch).**

$$l(D) - l(K_C - D) = \deg(D) + 1 - g$$

where $g$ is defined by the equation $\deg(K_C) = 2g - 2$.

We will prove this with a mix of algebra and analysis, following Mumford’s *Algebraic Geometry I: Complex Projective Varieties*.

**A Plausibility Argument.** A (rational) differential $\omega \in \Omega(C)$ has a well-defined notion of a *residue* at each point $p \in C$. If $z$ is a local (analytic) coordinate near $p$, with $z = 0$ at $p$, and if:

$$\omega = (b_{-d}z^{-d} + \cdots + b_{-1}z^{-1} + b_0 + \cdots)dz$$

then

$$\text{res}_p(\omega) = \frac{1}{2\pi i} \int_{\gamma} \omega = b_{-1}$$

where $\gamma$ is an oriented (small) loop around $p$. This is remarkable, since it tells us that the coefficient of $z^{-1}$ is intrinsic to the differential, and does not depend upon the choice of analytic local coordinate.

If $D = \sum d_ip_i \geq 0$ and $z_i$ are local coordinates near $p_i$, we may let:

$$V = \{ a_{i,-d_i}z_i^{-d_i} + \cdots + a_{i,-1}z_i^{-1} \}_{i=1}^n$$

be the vector space of “potential Laurent parts” of a function $f \in L(D)$. The *Mittag-Leffler problem* asks when a potential Laurent part is the collection of Laurent tails of some $f \in L(D)$. Notice that if $f_1, f_2$ both solve the same Mittag-Leffler problem, then:

$$f_1 - f_2$$

is holomorphic everywhere, hence constant so the solutions are unique, up to addition of a constant.

With residues, we see that the regular differentials on $C$ produce conditions on solvability of the Mittag-Leffler problem. Specifically:

$$\sum_{i=1}^n \text{res}_{p_i}(f\omega) = 0$$

for any $f \in L(D)$ and $\omega \in \Omega[C]$ by Stokes’ Theorem.
This is a linear condition on Laurent parts in \( V \). If
\[
\{ (b_{i,0} + \cdots + b_{i,d_i-1}z_i^{d_i-1})dz_i \}_{i=1}^n
\]
are the initial parts of the differential \( \omega \) at each \( p_i \), then:
\[
\sum_{i=1}^n \text{res}_{p_i}(f\omega) = \sum_{i=1}^n (a_{i,-d_i}b_{i,d_i-1} + \cdots + a_{i,-1}b_{i,0})
\]
and this is only identically zero if all the initial parts of \( \omega \) are zero, i.e. \( \omega \in L(K_C) \) fails to impose a linear condition on \( V \) iff \( \omega \in L(K_C - D) \). Thus:
\[
\dim(L(D)/C) \leq \dim(V) - \dim(L(K_C)/L(K_C - D))
\]
which (since \( \dim(V) = \deg(D) \)) gives an inequality:
\[
l(D) - l(K_C - D) \leq \deg(D) + 1 - l(K_C)
\]
We will see that the inequality is an equality, and that \( l(K_C) = g \) which will give us the Riemann-Roch Theorem.

**A Reduction.** Suppose for every \( D \) there is a divisor \( E \) such that:

(i) \( E - D \geq 0 \) (i.e. \( D \) “is contained in” \( E \)), and

(ii) the Riemann-Roch Theorem holds for \( E \).

Then the Riemann-Roch Theorem holds for every \( D \).

**Proof.** If \( D = \sum d_ip_i \) and \( E = \sum e_ip_i \) with \( e_i \geq d_i \), let:
\[
V = \{ (a_{i,-e_i}z_i^{-e_i} + \cdots + a_{i,-d_i+1}z_i^{-d_i+1}) \}_{i=1}^n
\]
be the space of Laurent tails “between” an \( f \in L(D) \) and a \( g \in L(E) \). Then the natural “Laurent tail map” \( T \) has kernel \( L(D) \):
\[
0 \rightarrow L(D) \rightarrow L(E) \xrightarrow{T} V
\]
since a rational function in \( L(D) \) is a rational function in \( L(E) \) with no Laurent tail between \( D \) and \( E \).

Next, consider the residue pairing with a differential \( \omega \in L(K_C - D) \):
\[
\text{res} : V \times L(K_C - D) \rightarrow \mathbb{C}; \ v \times \omega \mapsto \sum_i \text{res}_{p_i}(v \cdot \omega)
\]
where \( v \cdot \omega \) is understood to be the set of \( n \) locally defined differentials:
\[
\{ a_{i,-e_i}z_i^{-e_i} + \cdots + a_{i,-d_i+1}z_i^{-d_i+1} \} \cdot \omega
\]
near each point \( p_i \in C \). Then \( \text{res}(v, \omega) = 0 \ \forall v \in V \iff \omega \in L(K_C - E) \) since only a differential with zeroes of order \( e_i \) or more at each \( p_i \) will produce a zero overall residue (as in the plausibility argument above).
But if \( g \in L(E) \), then
\[
\sum_{i=1}^{n} \text{res}_{p_i}(T(g) \cdot \omega) = \sum_{i=1}^{n} \text{res}_{p_i}(g \omega) = 0
\]
by Stokes’ theorem, so the image of \( T \) pairs with zero against any differential in \( L(K_C - D) \). This gives us the following sequence of vector spaces that is exact everywhere except (possibly) the middle:
\[
0 \to L(E)/L(D) \xrightarrow{T} V \xrightarrow{\text{res}} L(K_C - D)^*/L(K_C - E)^* \to 0
\]
where the latter map \( \text{res} \) is defined via the residue pairing.

It follows that:
\[
\deg(E) - \deg(D) = \dim(V) \geq l(E) - l(D) + l(K_C - D) - l(K_C - E)
\]
and therefore that if:
\[
l(E) - l(K_C - E) = \deg(E) + 1 - g
\]
then:
\[
l(D) - l(K_C - D) \geq \deg(D) + 1 - g
\]

On the other hand, we may apply the same argument with the roles of \( D \) and \( K_C - D \) reversed (containing \( K_C - D \) in a divisor \( E \) for which Riemann-Roch holds) to get:
\[
l(K_C - D) - l(D) \geq \deg(K_C - D) + 1 - g = (2g - 2) - \deg(D) + 1 - g
\]
which gives us the opposite inequality and hence equality. \( \square \)

Now we get to the heart of the matter by connecting linear series with the homogeneous coordinate ring:
\[
R = \mathbb{C}[x_0, \ldots, x_n]/I(C) \text{ of the embedded curve } C \subset \mathbb{P}^n
\]

**Observation.** Each homogeneous \( F_d \in R_d \) defines an effective divisor \( E_d \) on \( C \) via the following:
\[
\text{ord}_p(F_d) := \text{ord}_p(F_d/G) \text{ for any } G \in R_d \text{ with } G(p) \neq 0 \text{ and}
\]
\[
E_d := \text{div}(F_d) = \sum_{p \in C} \text{ord}_p(F_d) \cdot p
\]
and notice that if \( \text{div}(F_d') = E_d' \), then \( \text{div}(F_d'/F_d) + E_d = E_d' \) so all such divisors are linearly equivalent. We get injective maps:
\[
f_d : R_d \to L(E_d); \ G \mapsto G/F_d \text{ for all } d \geq 0
\]

**Proposition 7.1.** There is a \( d_0 \) such that \( f_d \) is surjective for all \( d \geq d_0 \).

In other words, there is a \( d_0 \) such that:

\((*)\) If \( d \geq d_0 \) and \( D \in |E_d| \) then \( D = \text{div}(G) \) for some \( G \in R_d \).
This will show, in particular, that \( l(E_d) = \dim(R_d) \) for all \( d \geq d_0 \) is computed by the Hilbert polynomial of \( R \).

Assume \( C \) does not lie in any of the coordinate hyperplanes, and let \( H_i = \text{div}(x_i) \), and consider any of the exact sequences:

\[
0 \to R_{d-1} \xrightarrow{x_i} R_d \xrightarrow{\pi^*} (R_{H_i})_d
\]

Then:

\[
R_{H_i} = \mathbb{C}[x_0, ..., x_n]/\langle x_i, I(C) \rangle
\]

has constant Hilbert polynomial \( \delta = \text{deg}(H_i) \), the degree of \( C \subset \mathbb{CP}^n \), so the Hilbert polynomial of \( R \) is \( h_R(d) = d\delta + c \) for some constant \( c \), and therefore (after possibly raising the value of \( d_0 \)),

\[
l(E_d) = d\delta + c = \text{deg}(E_d) + c \text{ for all } d \geq d_0
\]

since \( \text{deg}(E_d) = d \cdot \text{deg}(H_i) \). Also, since \( \text{deg}(E_d) > 2g - 2 \) for large \( d \), if we can additionally show that

\[
c = 1 - g
\]

then we have the Riemann-Roch theorem for all \( E = E_d \) and \( d \geq d_0 \).

To prove the Proposition, we will use three tools:

(a) **Noether Normalization.** Under a “general” projection:

\[
\pi_V : C \to \mathbb{CP}^1
\]

(from a codimension two subspace \( V \subset \mathbb{C}^{n+1} \), \( R \) is a finitely generated graded module over the homogeneous coordinate ring \( \mathbb{C}[z_0, z_1] \) of \( \mathbb{CP}^1 \) (with \( z_0 = \sum a_i x_i \) and \( z_1 = \sum b_i x_i \)).

(b) **Nullstellensatz.** If \( \phi \in K(C) \) is regular at all points of \( C \cap U_i \), i.e. if \( \phi \in \mathcal{O}_{C,p} \) for all \( p \in C \cap U_i \), then:

\[
\phi \in \mathbb{C}[C \cap U_i] = \left\{ \frac{F}{x_i^N} \mid F \in R_N \right\} \subset K(C)
\]

is in the coordinate ring of the affine curve \( C \cap U_i \subset U_i = \mathbb{C}^n \).

(c) Let \( M \) be a finitely generated graded torsion-free module over \( \mathbb{C}[z_0, ..., z_r] \), and let \( M_K \) be the localization of \( M \) with respect to the field \( \mathbb{C}(z_0, ..., z_r) \). Then there is a \( d_0 \) such that, for all \( d \geq d_0 \), if

\[
m \in M_K \text{ and } z_0^N m, ..., z_r^N m \in M_{N+d} \text{ for some } N
\]

then

\[
m \in M_d
\]
Let us assume (a)-(c) for now and use them to prove the Proposition. Suppose $D \in |E_d|$. Since $|E_d| = |dH_i|$ for each $i$, there are rational functions $\phi_i \in K(C)$ such that:

$$\text{div}(\phi_i) = D - dH_i$$

from which we conclude that $\text{div}(\phi_i/\phi_j) = \text{div}(x_i^d/x_j^d)$ hence $\phi_i/\phi_j$ is a constant multiple of $x_i^d/x_j^d$. After multiplying each $\phi_i$ by a suitable constant, we can arrange for:

$$F = \phi_0 x_0^d = \phi_1 x_1^d = \cdots = \phi_n x_n^d \in K(R)$$

and moreover $\text{div}(F) = D$, so we need to show that there is a $d_0$ such that $F \in R_d$ if $d \geq d_0$. We now invoke (a)-(c) as follows:

(a) Choose a generic projection so that $R$ is a finitely generated (graded) module over $\mathbb{C}[z_0, z_1]$. It follows that the field of fractions $K(R)$ agrees with the localization $R_K$ of $R$ at $K = \mathbb{C}(z_0, z_1)$, and in particular that if $m_1, ..., m_k$ generate $R$ as a $\mathbb{C}[z_0, z_1]$-module, then a subset of the $m_i$'s are a basis for $R_K$ as a vector space over $\mathbb{C}(z_0, z_1)$.

(b) Since $\text{div}(\phi_i) = D - dH_i$ only has negative coefficients at points of $C - C \cap U_i$, it follows that the $\phi_i \in \mathcal{O}_{C,p}$ for all $p \in C \cap U_i$, hence $\phi_i \in \mathbb{C}[C \cap U_i]$, and so for some $N$ and all $i$, we have $x_i^N \phi_i \in R_N$, so:

$$x_0^N F, ..., x_n^N F \in R_{N+d}$$

(c) It follows that for $N' > (n+1)(N-1)$, every monomial of degree $N'$ in $x_0, ..., x_n$ has degree $N$ or more in some $x_i$, hence:

$$z_0^{N'} F, z_1^{N'} F \in R_{N'+d}$$

(since $z_0 = \sum_{i=0}^{n} a_i x_i$ and $z_1 = \sum_{i=0}^{n} b_i x_i$). Thus there is a $d_0$ such that, for all $d \geq d_0$, we have $F \in R_d$, proving Proposition 7.1.

\textbf{Computation of the Constant.} Consider a projection:

$$\pi_V : C \to \mathbb{CP}^2$$

for $V \subset \mathbb{C}^{n+1}$ a general subspace of codimension three.

\textbf{Claim.} The image of $\pi_V$ is a nodal curve, and $\pi_V$ "resolves" the nodes.

\textbf{Sketch of the Proof.} Any projection $\pi_V$ is a composition:

$$\pi_q \circ \cdots \circ \pi_{q_1} : C \to \mathbb{CP}^2$$

of projections from points (in successively smaller projective spaces). So it suffices to show that for general choice of $q \in \mathbb{CP}^n$,

$$\pi_q : C \to \mathbb{CP}^{n-1}$$

is an embedding if $n > 3$ and a resolution of a nodal curve if $n = 3$. 
This is accomplished with a dimension count. Namely, if \( C \subset \mathbb{CP}^n \), consider the “secant” mapping to the Grassmannian:
\[
s : C \times C \to Gr(2, \mathbb{C}^{n+1}); \quad (p_1, p_2) \mapsto \overline{p_1p_2}
\]
and the incidence correspondence:
\[
\begin{array}{ccc}
\mathbb{CP}^n & \uparrow \pi_1 & Gr(2, \mathbb{C}^{n+1}) \\
\downarrow \pi_2 \\
Fl(1, 2, \mathbb{C}^{n+1})
\end{array}
\]
from the flag manifold. Then the locus of points \( q \in \mathbb{CP}^n \) that lie on a secant line of \( C \) is:
\[
\pi_1(\pi_2^{-1}(s(C \times C)))
\]
which has dimension at most 3, and therefore cannot fill \( \mathbb{CP}^n \) if \( n > 3 \) and the projection is injective if \( q \) lies on no secant line. Similarly, for the tangent map \( t : C \to Gr(2, \mathbb{C}^{n+1}) \):
\[
\pi_1(\pi_2^{-1}t(C))
\]
is (at most) two-dimensional, and it follows that the projection is an immersion if \( q \) lies on no tangent line. This shows that the general projection to \( \mathbb{CP}^3 \) (or higher) is an embedding, and moreover a general projection to \( \mathbb{CP}^2 \) is an immersion. To see that all the singular points of the latter are nodes requires a more delicate but similar analysis of the loci of trisecant lines and “parallel tangent” lines to a curve in \( \mathbb{CP}^3 \).

We are now ready to invoke the genus computation of §6. Namely,
\[
g = \left( \frac{\delta - 1}{2} \right) - \nu
\]
where \( 2g - 2 = \deg(K_C) \) and \( \nu \) is the number of nodes in \( \pi_V(C) \subset \mathbb{CP}^2 \).

Finally, we need a Hilbert polynomial computation:
\[
h_{\pi_V(C)}(d) = \binom{d + 2}{2} - \binom{d + 2 - \delta}{2} = d\delta + 1 - \binom{\delta - 1}{2}
\]
because \( \pi_V(C) \subset \mathbb{CP}^n \) is a plane curve of degree \( \delta \), and the Hilbert polynomial of \( C \subset \mathbb{CP}^m \) can be shown to satisfy:
\[
h_C(d) = h_{\pi_V(C)}(d) + \nu
\]
giving the desired computation of the constant term.

Finally, Riemann-Roch follows from the reduction since every divisor \( D \) on \( C \) can be evidently contained in a divisor of the form \( E_d \) for any sufficiently large value of \( d \)! \( \square \)