6|30-4

Normalization for:

$$k[x,y]/\langle xy-1\rangle$$

Not finite module $/k[x]$

$$k[x] \subsetneq x^{-1}k[x] \subseteq x^{-2}k[x] \subseteq \dots \subseteq k[x, x^{-1}]$$

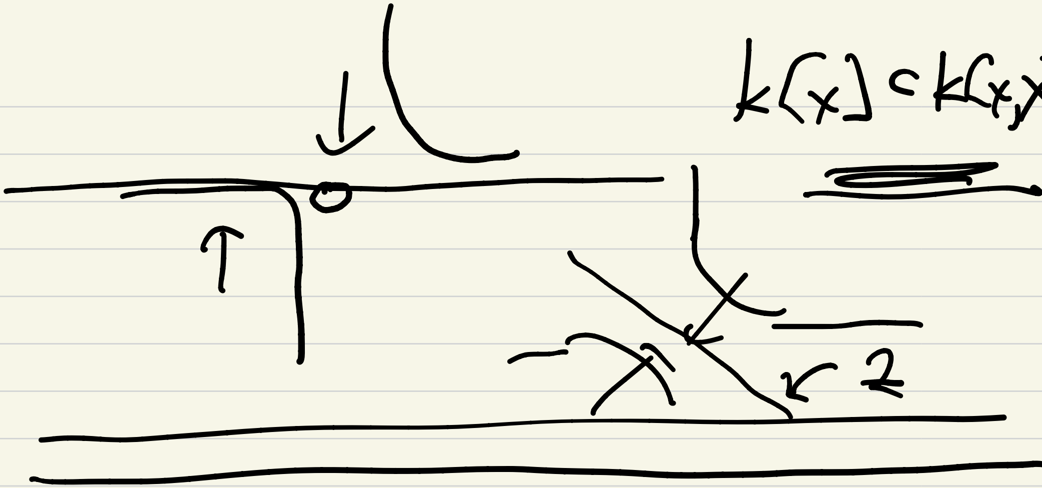
Let $z = x - y$; $x = y + z$

$$k[z,y]/(y+z)y-1$$

$$y^2 = 1 - zy \quad = \quad y^2 + zy - 1$$

Gen. by $1, y$ as a $k[z]$-mod

$k[x] \subset k[x, x^{-1}]$



<u>**Nullstellensatz**</u>: If k is infinite and $m \subset k[x_1, \dots, x_n]$ is a maximal ideal, then

$$k \subseteq k[x_1, \dots, x_n] \longrightarrow K = k[x_1, \dots, x_n]/m$$

is a <u>finite</u> field extension.

(If $k = \bar{k}$, then $k = K$ and $\rightarrow m = m_a$ for some $a \in k^n$)

<u>Pf</u>: If not, then

$$k \subset k[x_1,\dots,x_n]/m = K$$

is a field <u>and</u> finitely generated as a k-algebra, hence

$\exists\ y_1,\dots,y_d$ s.t. finite

$$k[y_1,\dots,y_d] \subseteq K \text{ is}$$

a finite module (<u>Noether Norm</u>)

would be a finite

<u>Nonsense if $d>0$</u>

$$\underline{k[y_1,\dots,y_d]} \overset{\cancel{\text{finite}}}{\subset} \underline{k[y_1,\dots,y_d,y_d^{-1}]} \subseteq \underline{K}$$

Corollary:

$$\boxed{k = \bar{k}}$$

If $f_1, \dots, f_m \in k[x_1, \dots, x_n]$

and $X(f_1, \dots, f_m) = \emptyset$, then

$\exists\, g_1, \dots, g_m \in k[x_1, \dots, x_n]$ s.t.

$$1 = \sum g_i f_i \quad *$$

Pf: If $X(f_1, \dots, f_m) \neq \emptyset$,

then $\langle f_1, \dots, f_m \rangle \subsetneq m_a \;\; \forall a$

$\Rightarrow \langle f_1, \dots, f_m \rangle \ni 1$.

Given $I \subset A$ (com. w/ 1)

Def:

$$\operatorname{rad}(I) = \{a \in A \mid a^n \in I \text{ for some } n\}.$$

Note:

$$I \subseteq \operatorname{rad}(I)$$

and

$$\operatorname{rad}(\operatorname{rad}(I)) = \operatorname{rad}(I).$$

Cor: Given $I \subseteq k[x_1, \dots, x_n]$,

then $I(X(I)) = \operatorname{rad}(I)$.

Pf:

<u>To</u> <u>show</u>: $I(X(I)) \subseteq \mathrm{rad}(I)$

($\supseteq$ obvious)

Suppose $f \in I(X(I))$

Let $I = \langle f_1, \dots, f_m \rangle$.

<u>Trick</u>: Reduce to <u>Null</u> by:

$$J := \langle f_1, \dots, f_m, f x_{n+1} - 1 \rangle \subseteq k[x_1, \dots, x_{n+1}]$$

Then $X(J) = \emptyset \subseteq \underline{k^{n+1}}$

$$(f_1(a), \dots, f_m(a) = 0 \Rightarrow f(a) = 0 \Rightarrow f(a, x_{n+1}) \cdot x_{n+1}$$

$$= f(a) \cdot x_{n+1} = 0$$

$$\Rightarrow f \cdot x_{n+1} - 1 \neq 0$$

Then by Null $\quad J \subseteq k[x_1, \dots, x_{n+1}]$

$$1 = \sum g_i(x_1, \dots, x_{n+1}) f_i$$

$$+ g(x_1, \dots, x_{n+1}) \cdot (f \cdot x_{n+1} - 1)$$

Formally replace x_{n+1} with $1/f$

$$1 = \sum g_i\left(x_1, \dots, x_n, \frac{1}{f}\right) \cdot f_i$$

Multiply by $f^{N} \leftarrow$ large

$$f^N = \sum h_i \cdot f_i \in I \quad \square.$$

Cor: $\{\text{geometric ideals } I(X)\}$

$$\updownarrow$$

$\{\text{radical ideals } I \subseteq k[x_1, \ldots x_n]\}$

Pf: (1) $I(X)$ are all radical

(2) radical ideals are geom.

$$I(X(I)) = I$$

(3) If $I \neq J$ (radical)

$$\Longleftrightarrow \quad X(I) \neq X(J)$$

(Hit with I)

Ex: (of a radical ideal)

Prime ideals P

Ex: Principal ideals in

$k[x_1, \dots, x_n]$ ✓(UFD)

$$I = \langle f \rangle \overset{\text{rad}}{\Longleftrightarrow} m_1 = \dots = m_k = 1$$

$$f = f_1^{m_1} \cdots f_k^{m_k} \quad \overset{\text{prime}}{\Longleftrightarrow} \quad f = f_i \text{ some } i.$$

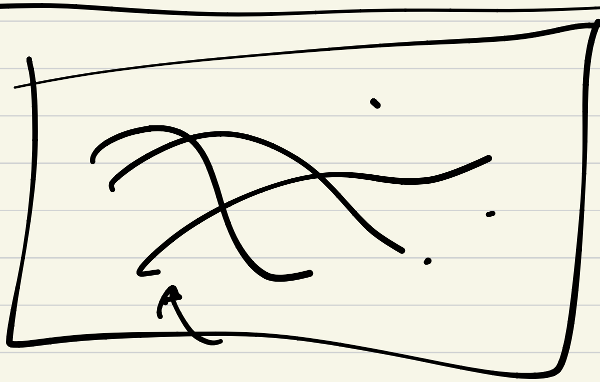
(prime factorization of f).

E.g. $\langle xy \rangle$ radical, not prime.

$\langle x_1 x_4 - x_2 x_3 \rangle$ prime.

$\mathcal{P}$ $\qquad\qquad$ $X(\mathcal{P})$

Prime ideals $\Longleftrightarrow$ irreducible closed sets



Alg set

Feature of the Zariski topology

$$(X \supseteq X_1 \supseteq \dots \supseteq X_n = X_{n+1} = \dots)$$

Def: X is irred. $\Longleftrightarrow$

$\nexists\ X = X_1 \cup X_2$ s.t. $X_1 \subsetneq X$, $X_2 \subsetneq X$, X_1, X_2 closed.

It ~~...~~ from des. chain cond. that every closed set $X = X(I)$ is a finite union of irreducible closed sets.

(Related to primary decomp.)

$$I = P_1 \cap \cdots \cap P_m$$

Claim: A radical ideal I is prime $\Leftrightarrow$ $X(I)$ is irred.