Algebraic Geometry I (Math 6130)

Utah/Fall 2020

5. More Projective Varieties.

We prove that projective varieties are proper and also discuss various equivalent formulations of the **dimension** of a variety, one of which is the degree of the Hilbert polynomial, when X is projective.

Theorem 5.1. $\mathbb{P}^n_k$ is a proper variety for all n.

Proof. Let X be a prevariety with affine open cover $\{Y_i\}$. Then

$$\pi_X : X \times \mathbb{P}^n_k \to X$$

is a closed map if each projection $\pi_{Y_i} : Y_i \times \mathbb{P}^n_k \to Y_i$ is a closed map, and if $Y \subset \mathbb{A}^m_k$ is a closed subvariety, then π_Y is a closed map if $\pi_{\mathbb{A}^m_k} : \mathbb{A}^m_k \times \mathbb{P}^n_k \to \mathbb{A}^m_k$ is closed. So we are reduced to showing the projections:

$$\pi : \mathbb{A}^m_k \times \mathbb{P}^n_k \to \mathbb{A}^m_k \text{ are closed maps for all } m \text{ and } n$$

In other words, we need to show that if $Z \subset \mathbb{A}^m_k \times \mathbb{P}^n_k$ is a closed subset, then:

$$X(I(\pi(Z))) = \pi(Z) \subset \mathbb{A}^m_k$$

since this is the defining property of a closed subset of $\mathbb{A}^m_k$.

Step 1. Consider the $k[y_1, ..., y_m]$-algebra:

$$A_\bullet = k[y_1, ..., y_m] \otimes_k S_\bullet$$

of polynomials in $x_0, ..., x_n$ (graded by degree) with coefficients in $k[y_1, ..., y_m]$. Then a homogeneous polynomial $F \subset A_d$ determines a well-defined subset:

$$X(F) = \{((a_1, ...a_n), (b_0 : .... : b_n)) \mid F(a_1, ..., a_n, b_0, ..., b_n) = 0\} \subset \mathbb{A}^m_k \times \mathbb{P}^n_k$$

that is a closed subset of the product prevariety, since for each $U_i \subset \mathbb{P}^n_k$,

$$X(F) \cap (\mathbb{A}^m_k \times U_i) = X(f) \subset \mathbb{A}^m_k \times U_i = \mathbb{A}^{m+n}_k$$

for $f = F/x_i^d = F(y_1, ..., y_n, \frac{x_0}{x_i}, ..., \frac{x_n}{x_i})$. Thus as in the case of $\mathbb{P}^n_k$, homogeneous ideals $I \subset A_\bullet$ determine closed subsets $X(I) \subset \mathbb{A}^m_k \times \mathbb{P}^n_k$. In fact, we claim that this property **characterizes** closed subsets $Z \subset \mathbb{A}^m_k \times \mathbb{P}^n_k$. Indeed, given Z, we define a homogeneous ideal I with:

$$I_d = \{F \in A_d \mid \frac{F}{x_i^d} \in I(Z \cap (\mathbb{A}^m_k \times U_i)) \text{ for all } i\}$$

Then $Z \subset X(I)$ since every such homogeneous polynomial F vanishes of Z, by construction. On the other hand, if $p = ((a_1, ..., a_m), (b_0 : ... : b_n)) \notin Z$ but $p \in \mathbb{A}^m_k \times U_i$, then there is an $f \in I(Z \cap (\mathbb{A}^m_k \times U_i))$ such that $f(p) \neq 0$, and:

$$x_i^d f = F \in I_d \text{ for all sufficiently large values of } d$$

so $p \notin X(F)$, and we conclude that $Z = X(I)$ for this homogeneous ideal I.

Step 2. Notice that $I_0 = I(\pi(Z))$, by definition. We claim that $\pi(Z) = X(I_0)$.

To prove this, we will use the full homogeneous ideal I of Z and:

Nakayama's Lemma. If A is a commutative ring with 1, M is a finitely generated A-module, and $I \subseteq A$ is an ideal such that $IM = M$ then $aM = 0$ for some $a \in 1+I$.

Proof. Let $m_1, ..., m_n$ generate M. By assumption, we can solve:

$$m_i = \sum_{j=1}^{n} b_{ij} m_j \text{ for a matrix } B = (b_{ij}) \text{ of elements } b_{ij} \in I$$

Then the matrix $I_n - B$ annihilates all elements $m = \sum a_i m_i \in M$ and then by Cramer's rule, $a = \det(I_n - B)$ also annihilates M, and has the desired form. $\square$

Proof of Step 2. Let $p \notin \pi(Z)$. Our goal is to find an $f \in I_0$ so that $f(p) \neq 0$.

Let $m_p \subset k[y_1, ..., y_m]$ be the associated maximal ideal, and consider the pair of homogeneous ideals I (from Step 1), and $m_p \otimes S_\bullet = m_p A_\bullet$. Since:

$$X(I) = Z \text{ and } X(m_p \otimes S_\bullet) = \pi^{-1}(p)$$

are **disjoint** closed subsets of $\mathbb{A}^m_k \times \mathbb{P}^n_k$, we may conclude that:

$$(*)\ I_d + m_p \cdot A_d = A_d$$

for sufficiently large values of d (as in Lemma 4.4). By Nakayama's Lemma applied to the finitely generated $k[y_1, ..., y_m]$-module A_d/I_d and ideal m_p, we conclude that there is an $f \in 1 + m_p$ so that $f \cdot A_d/I_d = 0$, i.e. $f \cdot A_d \subset I_d$. Thus:

$$f x_i^d \in I_d \text{ for all } i = 0, ..., n$$

and then it follows that $f \in I_0$ and $f(p) \neq 0$, as desired. $\square$

Corollary 5.2. All projective varieties are proper.

Example. Consider the *universal hypersurface* degree d hypersurface in $\mathbb{P}^n_k$ defined by the bihomogeneous polynomial inserting variables in place of the coefficients:

$$F = \sum_{|I|=d} y_I x_0^{i_1} \cdots x_n^{i_n}$$

defining a closed (projective) subvariety $X(F) \subset \mathbb{P}_k^{\binom{n+d}{n}-1} \times \mathbb{P}^n_k$.

This is the "universal family" of hypersurfaces of $\mathbb{P}^n_k$ in the sense that:

$$\pi^{-1}(\cdots : p_I : \cdots) = X(\sum_I p_I x_I) \subset \{p\} \times \mathbb{P}^n_k$$

is the hypersurface with coefficients p_I. Recalling that $(c_0 : ... : c_n) \in X(\sum_I p_I x_I)$ is a **singular** point if the gradient $\nabla(\sum_I p_I x_I)(c_0 : ... : c_n) = 0$ we can define the **relative** singular locus of the projection π as the closed subset:

$$Z = X(\langle \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \rangle) \subset \mathbb{P}_k^{\binom{n+d}{n}-1} \times \mathbb{P}^n_k$$

and then conclude that:

(*) The locus of non-singular hypersurfaces $X(F) \subset \mathbb{P}^n$ is open in $\mathbb{P}_k^{\binom{n+d}{n}-1}$.

The process of deriving equations for $\pi(Z) \subset X$ from equations for $Z \subset X \times \mathbb{P}^n_k$ is *elimination theory*, but merely just that the image is closed gives us information. For example, the Fermat hypersurface:

$$X(x_0^d + \cdots + x_n^d) \subset \mathbb{P}^n_k$$

is non-singular (if $\mathrm{char}(k)$ does not divide d), from which we conclude that the locus of non-singular hypersurfaces is not just non-empty, but also open (and dense!).

Consider now a **graded module** $M_\bullet$ over the polynomial ring S. That is,

$$M_\bullet = \bigoplus_{d\in\mathbb{Z}} M_d \quad \text{with} \quad S_d \cdot M_e \subset M_{d+e}$$

The finitely generated graded modules over $S_\bullet$ determine *coherent sheaves* on $\mathbb{P}^n_k$. This will be another conversion of an algebraic object to a geometric structure (coherent sheaves on $\mathbb{P}^n_k$ include vector bundles on closed subvarieties $X \subset \mathbb{P}^n_k$).

The twisted modules:

$$S_\bullet(k) := \bigoplus_{d\ge -k} S_{d+k}$$

are freely generated by $1 \in S(k)_{-k}$, and any homogeneous polynomial $F \in S_k$ determines a **graded** "multiplication by F" homomorphism:

$$S \stackrel{\cdot F}{\to} S(k); \quad G \mapsto FG$$

More generally, multiplication by F is a graded homomorphism:

$$S \otimes_S M_\bullet = M_\bullet \stackrel{\cdot F}{\to} S(k) \otimes_S M_\bullet = M_\bullet(k)$$

for any graded module $M_\bullet$, and its twist $M_\bullet(k) = \oplus M_{d+k}$.

Theorem 5.3. If $M_\bullet$ is a finitely generated graded module over $S_\bullet = k[x_0, ..., x_n]$, then the **Hilbert function**

$$h_M(d) = \dim_k M_d$$

agrees with the **Hilbert polynomial** $H_M(d)$, which is a polynomial of degree $\le n$, for all sufficiently large values of d.

Proof. The polynomials $H : \mathbb{Z} \to \mathbb{Z}$ of degree $\le n$ have a $\mathbb{Z}$-basis:

$$1 = \binom{d}{0}, d = \binom{d}{1}, ..., \binom{d}{n}$$

and if

$$H(d) = \sum_{i=0}^{n} a_i \binom{d}{i}, \text{ then } H(d+1) - H(d) = \sum_{i=0}^{n-1} a_{i+1} \binom{d}{i}.$$

Consider the graded homomorphism $\cdot x_n$ with kernel and cokernel N and L:

$$0 \to N \to M \stackrel{\cdot x_n}{\to} M(1) \to L \to 0$$

Then N and L are graded $k[x_0, ..., x_{n-1}]$ modules, and:

$$h_M(d+1) - h_M(d) = h_L(d) - h_N(d)$$

By induction, we may assume that $h_L(d)$ and $h_N(d)$ are polynomial functions of degree $\le n-1$ for large values of d, and then:

$$h_M(d+1) - h_M(d) = \sum_{i=0}^{n-1} b_i \binom{d}{n-1}$$

for large d, and it follows that:

$$h_M(d) = \text{constant} + \sum_{i=1}^{n} b_{i-1} \binom{d}{i}$$

i.e. h_M is a polynomial function of degree $\le n$ for large d. $\square$

Examples. (a) The Hilbert polynomial of $S(k)$ is:

$$H_{S(k)} = \binom{d+n+k}{n}$$

and the Hilbert function is:

$$h_{S(k)}(d) = \begin{cases} 0 \text{ for } d \le -k \\ \\ H_{S(k)}(d) \text{ for } d \ge -k-n \end{cases}$$

(b) Let $F \in S_k$, and let $A_\bullet = S/\langle F \rangle$. From the short exact sequence:

$$0 \to S(-k) \stackrel{\cdot F}{\to} S \to A_\bullet \to 0$$

we see that $H_{A_\bullet}(d) = H_S(d) - H_{S(-k)}(d)$, which is a polynomial of degree $n-1$.

(c) If $A_\bullet = k[x_0, ..., x_n]_\bullet/P$, then the Hilbert polynomial of $X(P) \subset \mathbb{P}^n_k$ is $H_A(d)$. It is not an invariant of the isomorphism class of variety X itself. E.g. $A_\bullet$ and $A_{m\bullet}$ yield isomorphic projective varieties with Hilbert polynomials $H_A(d)$ and $H_A(md)$.

Dimension. The dimension of a variety is its most basic invariant.

Definition 5.4. Let X be a variety over k. Then the dimension of X is:

$$\dim(X) = \text{tr deg}_k k(X)$$

where $k(X)$ is the field of rational functions on X.

Topology Detects Dimension. $\dim(X)$ is the length n of the longest chain:

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

of closed irreducible subsets of X. Thus it is a topological invariant of X.

Proof. Given a chain as above, choose an open affine $U \subset X$ with $U \cap X_0 \ne \emptyset$. Then $Y_i = X_i \cap U$ are a chain of irreducible closed subsets of U, and $\overline{Y}_i = X_i$. Since $k(X) = k(U)$, it suffices to prove this for $X = U$, an affine variety. If:

$$Z \subset X = \text{maxspec}(A)$$

is a closed subvariety, let $f \in A$ be a regular function with $Z \subset X(f)$. Then Z is contained in one of the irreducible components $Y \subset X(f)$. On the other hand, by the Krull Principal Ideal Theorem, tr $\deg_k k(Y) =$ tr $\deg_k k(X) - 1$.

By induction, then, for **any** closed, irreducible $Z \subset X$, there is a chain:

$$Z = Y^c \subset Y^{c-1} \subset \cdots \subset Y^1 \subset Y^0 = X$$

of irreducible closed subsets and regular functions $f_i \in k[Y^{i-1}]$ such that:

$$Y^i \text{ is a component of } X(f_i) \subset Y^{i-1} \text{ and } \dim(Y^i) = \dim(X) - i$$

In other words the **codimension** $\text{cod}_X(Y^i)$ of Y^i in X is i and in particular, every (zero-dimensional) point $x \in X$ has codimension equal to $\dim(X)$. $\square$

Example. (a) The dimension of $X \times Y$ is $\dim(X) + \dim(Y)$. If:

$$X_0 \subset X_1 \subset \cdots \subset X_n = X \text{ and } Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$$

are maximal chains in X and Y, repsectively, then:

$$X_0 \times Y_0 \subset X_0 \times Y_1 \subset \cdots \subset X_0 \times Y_m \subset X_1 \times Y_m \subset \cdots \subset X_n \times Y_m = X \times Y$$

is a maximal chain of closed irreducible subsets of the product.

(b) If X is affine and $Z \subset X$ is a closed subvariety of codimension c, then the regular functions $f_1, ..., f_c$ in the proof above lift to regular functions in $k[X]$ with the property that $Z \subset X$ is an irreducible component of $X(f_1, ..., f_c)$. It is tempting to conclude that if Z_1, Z_2 have codimension c_1, c_2 in X, then every component of $Z_1 \cap Z_2$ has codimension $\le c_1 + c_2$. But this is false:

(!) Consider the three-dimensional variety $X = X(x_0x_3 - x_1x_2) \subset \mathbb{A}^4_k$. Then:

$$Z_1 = X(x_0, x_1) \subset X \text{ and } Z_2 = X(x_2, x_3) \subset X$$

are two-dimensional closed subvarieties (planes) in X, and:

$$Z_1 \text{ is a component of } X(x_0) \subset X \text{ and } Z_2 \text{ is a component of } X(x_3) \subset X$$

These two planes intersect only at the origin, which has codimension three in X and in particular is **not** a component of:

$$X(x_0, x_3) \subset X, \text{which consists of two lines!}$$

Definition 5.5. If X is a variety and $Z = X(f_1, ..., f_c) \subset X$ is irreducible of codimension c, then Z is a (set-theoretic) **complete intersection** in X.

Example. The planes Z_1, Z_2 above are not complete intersections. There is no single function $g \in k[X]$ such that $Z_1 = X(g)$.

Proposition 5.6. If X is an affine variety, $Z = X(f_1, ..., f_c) \subset X$ is a complete intersection, and $Z' \subset X$ is a closed subvariety of codimension c', then

$$\text{codim}_Y(X) \le c + c' \text{ for all irreducible components } Y \subset Z \cap Z'$$

Proof. As remarked above, we may conclude that Z' is an irreducible component of $X(g_1, ..., g_{c'})$ for regular functions $g_i \in k[X]$. Unlike example (!) we may now also conclude from Krull's Theorem that the irreducible components of $Z \cap Z' = X(f_1, ..., f_c) \cap Z'$ are irreducible components of $X(g_1, ..., g_{c'}, f_1, ..., f_c) \subset X$, and therefore have codimension $\le c + c'$ in X. $\square$

Corollary 5.7. If $Z, Z' \subset \mathbb{A}^n_k$ are closed subvarieties of codimension c and c', then every component of $Z \cap Z'$ has codimension $\le c + c'$ in $\mathbb{A}^n_k$.

Proof. We use the fact that:

$$Z \cap Z' = (Z \times Z') \cap \Delta \subset \mathbb{A}^n_k \times \mathbb{A}^n_k$$

The codimension of $Z \times Z'$ in $\mathbb{A}^{2n}_k$ is $c + c'$, and the codimension of Δ is n. Moreover, Δ is a complete intersection:

$$\Delta = X(y_1 - x_1, ..., y_n - x_n)$$

so the Proposition applies to the components of $Z \cap Z' = (Z \times Z') \cap \Delta$. . $\square$

Corollary 5.8. Closed subvarieties $Z_1, Z_2 \subset \mathbb{P}^n_k$ have a non-empty intersection when their codimensions satisfy $c_1 + c_2 \le n$.

Proof. The affine cones $C(Z_1), C(Z_2) \subset \mathbb{A}^{n+1}_k$ over Z_1 and Z_2 have the same codimensions c_1 and c_2 in $\mathbb{A}^{n+1}_k$ and $0 \in C(Z_1) \cap C(Z_2)$. But by Corollary 5.7, each component of $C(Z_1) \cap C(Z_2)$ (which is necessarily itself an affine cone), has codimension $\le c_1 + c_2 \le n$ in $\mathbb{A}^{n+1}_k$. Thus $0 \in C(Z_1) \cap C(Z_2)$ is contained in a component of positive dimension, and $Z_1 \cap Z_2 \subset \mathbb{P}^n_k$ is not empty. $\square$

Exercise. The affine cone $C(Z)$ over $Z \subset \mathbb{P}^n$ has dimension $\dim(Z) + 1$.

Examples. (a) Curves in $\mathbb{P}^2_k$ always intersect! (E.g. parallel lines meet at infinity). When counted with correct multiplicities, the number of points of $X(F) \cap X(G)$ for homogeneous polynomials F and G not sharing a common factor is:

$$\deg(F) \cdot \deg(G)$$

This is Bézout's Theorem. We will discuss it later.

(b) The non-empty intersection property of Corollary 5.8 is topological. Thus:

$$\mathbb{P}^{n+m}_k \text{ is not homeomorphic to } \mathbb{P}^n_k \times \mathbb{P}^m_k$$

because the latter fails Corollary 5.8. If $n \geq m$, then:

$$\mathbb{P}^n \times \{x_1\} \cap \mathbb{P}^n \times \{x_2\} = \emptyset \text{ for } x_1 \neq x_2 \in \mathbb{P}^m_k$$

but $\mathrm{codiim}_{\mathbb{P}^n \times \mathbb{P}^m}(\mathbb{P}^n \times \{x\}) = m$ and $2m \leq n + m$.

(c) If X is affine and $0 \neq f \in k[X]$, then **every** irreducible component of $X(f)$ has codimension one. This is not true of two or more functions. All we can say is that every component of $X(f_1, ..., f_c) \subset X$ has codimension c **or less**.

Hilbert Polynomials and Dimension. If $X = \mathrm{maxproj}(A_\bullet)$ is projective, then:

$$\dim(X) \text{ is the degree of the Hilbert polynomial } \mathrm{H}_{A_\bullet}(d)$$

Proof. We will use the closed embedding:

$$X \subset \mathbb{P}^n_k \text{ with } A_\bullet = S_\bullet / P$$

$(*)$ Projecting $\pi_p : \mathbb{P}^n - - > \mathbb{P}^{n-1}$ from a point $p \notin X$ restricts to a morphism:

$$\pi_p : X \to \mathbb{P}^{n-1}$$

that is finite onto its image. If $r = \dim(X)$, then projecting from $n - r$ (successive) points defines a morphism: $\pi : X \to \mathbb{P}^r_k$ which we can interpret as the projection from a linear projective subspace $\Lambda \subset \mathbb{P}^n_k$ of dimension $n - r - 1$ that does not intersect X. With a change of basis (of A_1), we may assume that the projection is:

$$\pi_\Lambda(a_0 : .... : a_n) = (a_0 : .... : a_r : 0 : ... : 0)$$

and when restricted to any of the affine open subsets U_i, $i = 0, ..., r$,

$$k[\frac{x_1}{x_i}, ..., \frac{x_d}{x_i}] \subset A_{(x_i)}$$

is a polynomial subring over which $A_{(x_i)}$ is a finite module (Noether Normalization). It follows that π is surjective and finite-to-one and that the Hilbert polynomial of:

$$A_\bullet / \langle x_1, ..., x_d \rangle = S_\bullet / \langle P, x_1, ..., x_d \rangle$$

is a constant $\delta > 0$, equal to the dimension of the k-algebra:

$$A_{(x_0)} / \langle \frac{x_1}{x_0}, ..., \frac{x_d}{x_0} \rangle$$

But it follows that:

$$\mathrm{H}_{A_\bullet}(d) = \delta \cdot \binom{d}{r} + \text{lower order}$$

and in particular that the Hilbert polynomial has degree r. □

Remark. The constant δ is the **degree** of the projective variety $X \subset \mathbb{P}^n$. Unlike the dimension, this is not an isomorphism invariant of the variety X, since, for example, the degree of $\mathbb{P}^1$ is one, but the degree of the conic $C \subset \mathbb{P}^2_k$ is two. Interestingly, though, we will see that the constant term of the Hilbert polynomial is an invariant.

Assignment 5.

1. Prove that the the affine cone $C(X)$ over a projective variety $X \subset \mathbb{P}^n$ satisfies:

$$\dim(C(X)) = \dim(X) + 1$$

2. A subvariety $X \subset \mathbb{P}^n_k$ is an ideal-theoretic complete intersection if the ideal: $I(X) = \langle F_1, ..., F_c \rangle$ is generated by $c = \mathrm{codim}_{\mathbb{P}^n}(X)$ homogeneous polynomials.

(a) Find homogeneous polynomials F, G of degrees two and three so that the twisted cubic $C \in \mathbb{P}^3_k$ is the set-theoretic complete intersection $C = X(F) \cap X(G)$.

(b) Find the leading coefficient of the Hilbert polynomial:

$$H_X(d) \text{ for } X = X(F_1, ..., F_c) \subset \mathbb{P}^n$$

of an ideal-theoretic complete intersection of polynomials of degrees $d_1, ..., d_c$.

(c) Compute the degrees (and codimensions) of the rational normal curves in $\mathbb{P}^n$. Conclude that they are never ideal-theoretic complete intersections.

3. (a) Prove that if $f : \mathbb{P}^n \to \mathbb{P}^m$ is a morphism then there are $m+1$ homogeneous polynomials $F_0, ..., F_m \in S_d$ for some d such that:

$$f(x) = (F_0(x) : .... : F_m(x))$$

"on the nose." I.e. $F_i(x) \neq 0$ for some i for each $x \in \mathbb{P}^n$.

(b) Conclude that there are **no** morphisms $f : \mathbb{P}^n \to \mathbb{P}^m$ when $m < n$.

4. A **nonsingular curve** is a variety C such that:

$$\dim(C) = 1 \text{ and } \mathcal{O}_{C,p} \text{ is a discrete valuation ring}$$

for all points $p \in C$. Prove that every rational map:

$$f : C - - > \mathbb{P}^n$$

is a morphism, defined at all points of C.

5. (a) Compute the dimension of the Grassmannian $\mathrm{Gr}(m, n)$ of m planes in k^n.

The degeneracy loci:

$$D_r = \{A : k^m \to k^n \mid \mathrm{rk}(A) \leq r\} \subset \mathbb{A}^{mn}_k$$

are algebraic subsets. There is an "incidence correspondence" between D_r and $\mathrm{Gr}(m-r, m)$ defined by:

$$I = \{(A, \Lambda) \mid \Lambda \subset \ker(A)\} \subset \mathbb{A}^{mn}_k \times \mathrm{Gr}(m-r, r)$$

This is an algebraic subset of the product,

(b) Analyze the fibers $\pi_2^{-1}(\Lambda) \subset I$ of the projection $\pi_2|_I : I \to \mathrm{Gr}(m-r, r)$ and use your analysis to argue that I is a variety and find its dimension.

(c) Analyze the morphism $\pi_1 : I \to D_r$ and conclude that $\dim(I) = \dim(D_r)$. Verify that the codimension of D_r in $\mathbb{A}^{nm}_k$ is $(n-r)(m-r)$.

6. Comment on the following. If $X \subset \mathbb{P}^n$, then any rational map:

$$f : X - - > \mathbb{P}^m$$

is given by $F_0, ..., F_m \in S_d/P_d$ (P is the homogeneous ideal of X). Reembedding X in $\mathbb{P}^{\binom{n+d}{n}-1}$ (via the d-uple embedding), the forms $F_0, ..., F_m$ become *linear*, and f is the restriction of a **projection** (rational map) from $\mathbb{P}^{\binom{n+d}{n}-1}$ to $\mathbb{P}^m$.