Algebraic Geometry I (Math 6130)

Utah/Fall 2020

6. Maps.

We study some basic properties of rational maps and regular maps of varieties. Upper semi-continuity of discrete invariants is a prominent feature in algebraic geometry, and we will see that the fiber dimension of a regular map $f : X \to Y$ of varieties is an upper semi-continuous function $e : X \to \mathbb{Z}$. As an application, we prove Chevalley's Theorem that proper algebraic groups are abelian.

A **regular map** $f : X \to Y$ of varieties is a morphism of sheaved spaces, i.e. f is a continuous map with the property that:

$$f^*(\mathcal{O}_Y(U)) \subset \mathcal{O}_X(f^{-1}(U))$$

for all open subsets $U \subset Y$. In particular, if the image of f is **dense** in Y, then passing to direct limits, we get:

$$f^* : k(Y) \hookrightarrow k(X)$$

an injection of fields of rational functions. Such a regular map is called **dominant**.

Conversely, an inclusion of fields of rational functions:

$$i : k(Y) \hookrightarrow k(X)$$

determines a **dominant rational map** $f : X - - > Y$ (partially) defined as a map to each open affine subset $U = \text{maxspec}(k[y_1, ..., y_n/P) \subset Y$ by:

$$f(x) = (i(y_1)(x), ..., i(y_n)(x)) \in U \subset \mathbb{A}^n_k$$

when each $i(y_j) \in \mathcal{O}_{X,x}$. Intrinsically, if $x \in X$, then $f(x) = y$ provided that:

$$\mathcal{O}_{X,x} \cap i(k(Y)) = i(\mathcal{O}_{Y,y})$$

and f is undefined at x if there is no such $y \in Y$. A general rational map from X to Y is a dominant rational map to a closed subvariety $Z \subset Y$.

A rational map f is regular when restricted to a well-defined (maximal) open subset $V \subset X$, which is its **domain**. Alternatively, let:

$$Z = \overline{\Gamma}_f \subset X \times Y$$

be the closure of the **graph** of f (restricted to V). Via the projection to Y, we get: $\overline{f} := \pi_Y : Z \to Y$ which is a regular *extension* of the regular map $f|_V : V \to Y$. We say that $X \stackrel{\pi_X}{\leftarrow} Z \stackrel{\overline{f}}{\to} Y$ **resolves the indeterminacy** of the rational map f.

Example. The map $f : \mathbb{A}^{n+1} - - > \mathbb{P}^n_k$; $f(a_0, a_1, ..., a_n) = (a_0 : a_1 : ... : a_n)$ is defined away from the origin. The closure of the graph of f is:

$$Z = X(\langle x_i y_j - x_j y_i \rangle) \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$$

If $(b_0 : ... : b_n) \in \mathbb{P}^n_k$, then:

$$\overline{f}^{-1}(b_0 : ... : b_n) = \{((\lambda b_0, ..., \lambda b_n), (b_0 : ... : b_n)) \mid \lambda \in k\} \subset Z$$

and the first projection $\pi : Z \to \mathbb{A}^{n+1}_k$ satisfies:

$$\pi^{-1}(0, ..., 0) = \{(0, ..., 0)\} \times \mathbb{P}^n_k \subset Z \text{ and } \pi : (\pi^{-1}(V)) \to V \text{ is an isomorphism}$$

for $V = \mathbb{A}^{n+1}_k - 0$.

This generalizes to affine cones. If $P \subset k[x_0, ..., x_n]$ is a homogeneous prime ideal and $X = X(P) \subset \mathbb{P}^n_k$, then the rational map $f : C(X) - - > X$ defined as above is defined away from the origin, and

$$Z = \overline{\Gamma}_f = X(\langle P, x_i y_j - x_j y_i \rangle) \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$$

for the homogeneous ideal in $A = k[x_0, ..., x_n] \otimes_k S_\bullet$ generated by P (in A_0) and the $x_i y_j - x_j y_j$ (in A_1). The fibers of the map $\overline{f} : Z \to X$ are lines and the map $\pi : Z \to C(X)$ is an isomorphism away from $0 \in C(X)$ with fiber $\{0\} \times X$ over 0.

Next we ask "What do dominant regular maps look like?"

Definition 6.1. A dominant regular map $f : X \to Y$ is birational if the associated injective map on fields $f^* : k(Y) \to k(X)$ is an isomorphism. Equivalently, f is birational if it has a rational inverse (corresponding to the isomorphism $(f^*)^{-1}$).

Proposition 6.2. If $f : X \to Y$ is a birational **regular** map, then there is a nonempty open subset $U \subset Y$ such that the induced regular map:

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is an isomorphism of varieties (i.e. a *biregular* map).

Proof. First, we show that there are open subsets of X and Y such that:

$$f|_V : V \to U \text{ is an isomorphism}$$

Let $U_0 \subset Y$ and $V_0 \subset f^{-1}(U_0)$ be open affine subsets. Then $f|_{V_0} : V_0 \to U_0$ is a birational regular map of affine varieties, so $f^* : k[U_0] \to k[V_0]$ is an injective map of k-algebras that is an isomorphism on fields of fractions. If $k[V_0] = k[x_1, ..., x_n]/P$ and $(f^*)^{-1}(x_i) = g_i/h_i \in k(U_0)$, let $h = \prod h_i$. Then:

$$f^* : k[U_0]_h \to k[V_0]_{f^*(h)}$$

is an isomorphism of k-algebras, and so $f : V_0 - X(h) \to U_0 - X(f^*h)$ is the desired isomorphism. Next, let $U_1 = U_0 - X(f^*h)$ and $Z = f^{-1}(U_1) - (V_0 - X(h))$. Then Z is an algebraic set, which is a union of irreducible closed sets Z_i of dimensions $\dim(Z_i) < \dim(X)$. The closures of their images $\overline{f(Z_i)} \subset U_1$ are also irreducible, of dimension at most $\dim(Z_i)$, and so:

$$U = U_1 - \overline{f(Z)}$$

is nonempty, and satisfies the requirement of the Proposition. $\square$

Example. Apply the proof of the Proposition to the map:

$$f : \mathbb{A}^2_k \to \mathbb{A}^2_k;\ f(x_1, x_2) = (x_1, x_1 x_2)$$

This map collapses the y-axis to the origin and does not map to any points of the form $(0, y)$ with $y \neq 0$. The map $f^* : k[y_1, y_2] \to k[x_1, x_2]$ takes $y_1 \mapsto x_1$ and $y_2 \mapsto x_1 x_2$ so $(f^*)^{-1}(x_1) = y_1$ and $(f^*)^{-1}(x_2) = y_2/y_1$. Thus:

$$f^* : k[y_1, y_2]_{y_1} \to k[x_1, x_2]_{x_1 = f^* y_1}$$

is an isomorphism, which corresponds to the isomorphism:

$$f : \mathbb{A}^2_k - X(x_1) \to \mathbb{A}^2_k - X(y_1) \text{ of planes minus } y\text{-axes}$$

Definition 6.3. A dominant **rational** map of varieties is birational if it has an inverse rational map, or equivalently if the map on fields is an isomorphism.

By restricting to the open domain of a birational map $f : X -- > Y$ and applying the Proposition, we obtain an open subset $U \subset Y$ such that $f : f^{-1}(U) \to U$ is an isomorphism. In other words, inside the image of every birational map of varieties there is an open set that is isomorphic to its preimage. Notice that it isn't even a priori clear that the image of a dominant rational map must contain an open set (all we know from the definition is that the image is dense!). On the other hand, a birational regular map $f : X \to Y$ of projective varieties is surjective because projective varieties are proper, but such a map need not be an isomorphism.

For general dominant regular maps, we have the following:

Theorem 6.4. If $f : X \to Y$ is a dominant map, let $r = \dim(X) - \dim(Y)$. Then:

(a) (i) Every irreducible component of every fiber $f^{-1}(y)$ has dimension $\geq r$.

More generally,

(ii) If $Z \subset Y$ is a closed subvariety and $W \subset f^{-1}(Z)$ is an irreducible component that dominates Z, then $\dim(W) \geq r + \dim(Z)$.

(b) There is an open subset $U \subset f(X)$ of the image of f with the following "transverse" property:

(i) If $y \in U$ then every component of $f^{-1}(y)$ has dimension **equal to** r.

More generally,

(ii) If $Z \subset U$ is a closed subvariety then every irreducible component of $f^{-1}(Z)$ has dimension $r + \dim(Z)$.

Proof. It suffices to prove (a) replacing Y by an affine open subset $U \subset Y$ intersecting Z, and Z by $Z \cap U$ and W by $W \cap f^{-1}(U)$. Then as in §5, $Z \cap U \subset U$ is an irreducible component of $X(g_1, ..., g_c)$ for $c = \text{codim}_Z(Y)$ and $g_1, ..., g_c \in k[U]$. Shrinking U (and $Z \cap U$) further to exclude the other components, we may assume that $Z = X(g_1, ..., g_c) \subset U$ is a set-theoretic complete intersection. Then $W \subset f^{-1}(Z) = X(f^*g_1, ..., f^*g_c)$ is an irreducible component, and by Krull's Theorem, it follows that $\text{codim}_X W \leq c$, which gives (a).

For (b), we may also assume Y is affine (replacing it with an open affine subset), and we may also assume X is affine, since if (b) holds for each $f : V_i \to Y$ for an open cover $\{V_i\}$ of X, then it holds for X itself. So we consider:

$$f : X \to Y \text{ and } f^* : k[Y] \hookrightarrow k[X] \subset k(X)$$

with $k(Y) \subset k(X)$ a field extension of transcendence degree r. Consider:

$$k(Y) \subset A = k(Y) \otimes_{k[Y]} k[X]$$

This is a finitely generated domain over the field $k(Y)$ with fraction field $k(X)$, so by the Noether Normalization Theorem,

$$(**) \quad k(Y)[x_1, ..., x_r] \subset A$$

is a finitely generated module for some $x_1, ..., x_r \in A$, which we may take to be elements of $k[X]$ (clearing denominators). Now consider:

$$(*) \ k[Y][x_1, ..., x_r] \subset k[X]$$

If this were a finitely generated module, we'd have:

$$f : X \stackrel{g}{\to} Y \times \mathbb{A}^r_k \stackrel{\pi_Y}{\to} Y$$

a composition of a finite map and a projection, which is a surjective dominant map satisfying (b) with $U = Y$! We can't conclude $(*)$ is a finite module, but if $\phi \in k[X] \subset A$, then ϕ satisfies a monic polynomial relation:

$$\phi^n + p_1\phi^{n-1} + \cdots + p_n = 0 \text{ for } p_i \in k(Y)[x_1, ..., x_r]$$

since $\phi \in A$ is a finite module over $k(Y)[x_1, ..., x_n]$, and if $h \in k[Y]$ is chosen so that $hp_i \in k[Y][x_1, ..., x_r]$ for all i, then ϕ is finite over $k[Y]_h[x_1, ..., x_r]$. Since $k[X]$ is finitely generated as an algebra over $k[Y][x_1, ..., x_r]$, it follows by induction that:

$$(***) \ k[Y]_h[x_1, ..., x_r] \subset k[X]_h$$

is a finitely generated module for some $h \in k[Y]$, giving (b) for $U = U_h \subset Y$. □

Definition 6.5. An integer-valued function:

$$e : X \to \mathbb{Z}$$

is upper-semi continuous if $X_n = \{x \in X \mid e(x) \geq n\} \subset X$ are closed for all n.

Remark. If X is an irreducible Noetherian space, then the nested closed sets:

$$\cdots \supset X_{n-1} \supset X_n \supset \cdots$$

stabilize to the empty set, and e is bounded above. If e has a minimum value r, then $U = X - X_{r+1}$ is a dense open subset on which $e(x) \equiv r$ for all $x \in U$.

Corollary 6.6. If $f : X \to Y$ is a dominant regular map, then the function:

$$e : X \to \mathbb{Z}; \ e(x) = \max\{\dim(Z) \mid Z \subset f^{-1}f(x) \text{ and } x \in Z\}$$

is an upper-semi continuous function with minimum value $r = \dim(X) - \dim(Y)$.

Proof. This follows from the Theorem and induction on the dimension of Y. The minimum value $e(x) = r$ is taken on the open set $f^{-1}(U) \subset X$ and $X - f^{-1}(U)$ is a union of irreducible components Z_i dominating $\overline{f(Z_i)} = W_i \subset Y - U$. Then $X_k = \cup(Z_i)_k$ for all $k > r$ is a union of closed sets by the inductive assumption, since $\dim(W_i) < \dim(Y)$. □

An Application. An **algebraic group** G over k is a pointed variety $e \in G$ together with regular "multiplication" and "inverse" maps:

$$m : G \times G \to G \text{ and } i : G \to G$$

that make the points of G into a group with identity element e.

Examples of **affine** algebraic groups include:

$$\mathbb{G}_m := \text{maxspec}(k[x, x^{-1}]) = k^* \text{ with multiplication and}$$
$$\mathbb{G}_a := \text{maxspec}(k[x]) = k \text{ with addition}$$

and the matridx groups:

$$\text{GL}(n, k), \ \text{SL}(n, k), \ \text{PGL}(n, k), \ \text{SO}(n, k), \text{Sp}(2n, k)$$

that are not abelian groups when $n > 1$.

Theorem (Chevalley). If $e \in G$ is a proper algebraic group, then G is abelian.

Proof. Consider the regular conjugation map:

$$c : G \times G \to G; \ c(g, h) = ghg^{-1}$$

We may think of this as a *family* of regular maps indexed by $h \in G$:

$$c_h : G \to G; \ \ c_h(g) = ghg^{-1}$$

Note that $c_e : G \to G$ is the **constant** map to $e \in G$ and that in general c_h is the constant map (to h) if and only if h is in the center of G.

The Theorem follows from:

Proposition 6.7. If $f : X \times Y \to Z$ is a regular map of varieties and X is proper, then if $f_y : X \to Z$ is constant for some $y_0 \in Y$, it is constant for **all** $y \in Y$.

Proof. Consider the graph $\Gamma_f \subset X \times Y \times Z$ and the projection:

$$\pi : \Gamma_f \to Y \times Z \text{ with image } \pi(\Gamma_f) = W$$

Then W is a closed subvariety of $Y \times Z$ since X is proper, and the fibers of π are:

$$\pi^{-1}(y, z) = \{x \in X \mid f_y(x) = z\} \subset X \times \{y\} \times \{z\}$$

Thus to show that f_y is the constant map it suffices to prove that the dimension of (each component of) $\pi^{-1}(y, z) \subset \Gamma_f$ is greater than or equal to $\dim(X)$.

Since Γ_f is the graph of f, we have $\dim(\Gamma_f) = \dim(X) + \dim(Y)$, and then the desired inequality follows from Theorem 6.4(a) for **all** y if we can show that $\dim(W) \leq \dim(Y)$. But projecting once more:

$$p : W \to Y$$

and using the fact that $f_{y_0}(X) = z_0$ is the constant map, we have:

$$p^{-1}(y_0) = \{(y_0, z) \mid f_{y_0}(x) = z \text{ for some } x \in X\} = \{(y_0, z_0)\}$$

is a single point, and by Theorem 6.4 (a) again, we conclude that $\dim(W) \leq \dim(Y)$.

Example. Start with an affine cubic curve in Weierstrass form:

$$X = X(y^2 - x^3 - Ax - B) \subset \mathbb{A}^2_k$$

with nonzero discriminant and take the closure:

$$E = X \cup \{e\} = X(y^2z - x^3 - Axz^2 - Bz^2) \subset \mathbb{P}^2_k$$

where $e \in E$ is the single extra "point at infinity" $e = (0 : 1 : 0)$. Define the inverse:

$$i : E \to E \text{ by } i(a : b : c) = (a : -b : c)$$

(noting that the inverse fixes e), and then:

$$m = i \circ t : E \times E \to E$$

where $t(p, q) \in E$ is the "third point of intersection" of E with the line $l = \overline{pq}$. There are two important points to verify:

a) Why is the map t a regular map?

(b) Why is the group law defined this way *associative*?