Algebraic Geometry I (Math 6130)

Utah/Fall 2020

0. Preview

What is Algebraic Geometry?

The quick answer is that algebraic geometry is the study of the *geometry* of the loci of solutions of systems of *polynomial* equations:

$$f_1(x_1, ..., x_n) = \cdots = f_m(x_1, ..., x_n) = 0$$

with coefficients in a field k (or commutative ring A). These loci, denoted by:

$$X = X(f_1, ..., f_m) \subset k^n$$

are the (affine) *algebraic sets* associated to the polynomials $f_1, ..., f_m \in k[x_1, ..., x_n[$. It should be noted that the set $X = X(I)$ only depends on the *ideal* $I = \langle f_1, ..., f_m\rangle$ generated by $f_1, ..., f_m$ in the polynomial ring $k[x_1, ..., x_n]$. Algebraic sets are the closed sets of the *Zariski topology* on k^n, used to define the *sheaf of regular functions* on k^n. This information transforms the vector space k^n into the *affine variety* $\mathbb{A}^n_k$.

Until we develop the theory of schemes, we will restrict ourselves to fields k that are algebraically closed, in which the closed sets $X \subset \mathbb{A}^n_k$ are in bijection with the radical ideals $I \subset k[x_1, ..., x_n]$ (which are always finitely generated), and in particular the points of $\mathbb{A}^n_k$ are in bijection with the maximal ideals. Each closed set is the union of finitely many *irreducible* closed sets $X(P)$, for prime ideals $P \subset k[x_1, ..., x_n]$ (in the theory of schemes, these also define points of $\mathbb{A}^n_k$.)

An *affine variety* is a set with a topology and sheaf of regular functions that is isomorphic to some irreducible closed subset $V = X(P) \subset \mathbb{A}^n_k$ with the Zariski topology and sheaf of regular functions inherited from $\mathbb{A}^n_k$. Although they are defined via irreducible closed sets in $\mathbb{A}^n_k$, the affine varieties are a "good" basis of **open** sets of an *abstract variety* over k, in the same way that open balls are a "good" basis of open sets in a differentiable or analytic manifold. The sheaf of regular functions allows us to use algebra to define dimension, non-singularity, normality and other important geometric characteristics of a variety. A non-singular abstract variety over the field $\mathbb{C}$ of complex numbers is a complex analytic manifold, but one of the virtues of algebraic geometry is that abstract varieties can be defined over (algebraically closed) fields of any characteristic.

A *projective variety* is a particular example of an abstract variety, defined by systems of *homogeneous* polynomial equations. Projective varieties are *proper*, which makes them analogues of *compact* manifolds. This means that, as is the case with compact manifolds, we can use auxiliary geometric constructions (vector bundles and cohomology theories) to define numerical invariants that quantify geometric characteristics of X. The algebraic geometry analogues are (coherent) sheaves of modules over the sheaf of regular functions and their coherent sheaf cohomology (computed with Cech cohomology on open affine acyclic covers of X). The first concrete example of this is the *genus* of a one-dimension non-singular projective variety, defined via either the sheaf of differential one-forms on X (analogous to the cotangent bundle) or the *Euler characteristic* of the sheaf of regular functions. This gives an algebraic computation of the number of holes in X when $k = \mathbb{C}$, and X is a compact Riemann surface.

The Plan.

- §1. Algebraic sets.
- §2. Affine and quasi-affine varieties.
- §3. Abstract varieties
- §4. Projective and quasi-projective varieties.
- §5. Geometric features.
- §6. Divisors and Line Bundles
- §7. Differentials
- §8. Coherent and quasi-coherent sheaves.
- §9. Cohomology of coherent sheaves
- §10. Serre Duality