Algebraic Geometry I (Math 6130)

Utah/Fall 2020

2. Affine Varieties.

The space k^n with the Zariski topology supports a *sheaf of regular functions*. This information defines **affine n-space** $\mathbb{A}^n_k$ over the (algebraically closed) field k.

Definition 2.1. A **presheaf** $\mathcal{A}$ of commutative rings on a topological space X consists of the following information:

- A commutative ring $\mathcal{A}(U)$ with 1 attached to each open subset $U \subseteq X$.
- Restriction homomorphisms $\rho_{U,V} : \mathcal{A}(U) \to \mathcal{A}(V)$ of commutative rings with 1 attached to each inclusion $V \subseteq U$ of open sets.

such that $\rho_{U,U} = \text{id}$ and restriction homomorphisms compose, i.e.

$$\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W} \text{ whenever } U \subseteq V \subseteq W$$

Remark. In other words, a presheaf is a *contravariant functor* from the category of open subsets of X to commutative rings with 1 (and $\mathcal{A}(\emptyset)$ is the ring with $0 = 1$).

Example. (a) Fix a commutative ring A with 1 and let:

$$\mathcal{A}(U) = A \text{ and } \rho_{U,V} = \text{id}_A \text{ for all } U, V \neq \emptyset$$

This is the **constant** presheaf.

(b) The (infinitely) *differentiable functions* $f : U \to \mathbb{R}$ defined on the open subsets $U \subset M$ of a differentiable manifold M are a presheaf, denoted by $\mathcal{C}^\infty_M$ with restriction maps $\rho_{U,V}$ being the restriction of the domain: $\rho_{U,V}(f) = f|_V$. Note that the restrictions are typically not surjective ring homomorphisms.

Definition 2.2. A presheaf $\mathcal{A}$ is a **sheaf** if, in addition:

(i) If $\{U_\lambda\}$ is an open cover of U, and $a \in \mathcal{A}(U)$ and $\rho_{U,U_\lambda}(a) = 0$ for all λ, then:

$$a = 0 \in \mathcal{A}(U)$$

(and conversely, if $a = 0$, then evidently $\rho_{U,U_\lambda}(a) = 0$ for all λ).

(ii) If $\{U_\lambda\}$ is an open cover of U and $a_\lambda \in \mathcal{A}(U_\lambda)$ are given, satisfying:

$$\rho_{U_\lambda, U_\lambda \cap U_\mu}(a_\lambda) = \rho_{U_\mu, U_\lambda \cap U_\mu}(a_\mu) \text{ for all } \lambda, \mu$$

then there is an $a \in \mathcal{A}(U)$ such that $\rho_{U,U_\lambda}(a) = a_\lambda$ for all λ.

Remark. Using (i), one concludes that the $a \in \mathcal{A}(U)$ in (ii) is uniquely determined.

Examples. (a) Constant presheaves satisfy (i), but usually not (ii). If U, V are disjoint (non-empty) open subsets of X and $\mathcal{A}(U) = A$ is the constant presheaf, then U, V are an open cover of $U \cup V$, and if (ii) held, then

$$0 \in \mathcal{A}(U) = A \text{ and } 1 \in \mathcal{A}(V) = A$$

would be the two restrictions of a single $a \in \mathcal{A}(U \cup V) = A$, yielding $0 = a = 1$. All the (interesting) commutative rings with $0 \neq 1$ therefore give constant presheaves that are not sheaves. Note, however, that all pairs of nonempty open sets intersect in the Zariski topology on k^n, and in that case the constant presheaves are sheaves!

(b) The presheaf $\mathcal{C}^\infty_M$ of differentiable functions is a sheaf. Functions are zero if they are zero locally, and functions are defined if they are defined locally.

Definition 2.3. A rational function $\phi \in k(x_1, ..., x_n)$ is **regular** at $a \in k^n$ if

$$\phi = \frac{f}{g} \text{ for } f, g \in k[x_1, ..., x_n] \text{ and } g(a) \neq 0$$

Remark. The ring of functions that are regular at $a \in k^n$ is the **localization**

$$k[x_1, ..., x_n]_{m_a}$$

where the localization of a domain A at a prime ideal P is defined by:

$$A_P = \left\{ \frac{f}{g} \mid f \in A, g \in A - P \right\} / \sim$$

As in the field of fractions $k(A)$, the relation is given by $f/g \sim p/q$ if and only if $fq = pg$. In fact, the field of fractions $k(A)$ is the localization of A at the zero ideal.

Definition 2.4. The sheaf $\mathcal{O}_{\mathbb{A}_k^n}$ of **regular** functions on $\mathbb{A}_k^n$ is defined by:

$$\mathcal{O}_{\mathbb{A}_k^n}(U) = \bigcap_{a \in U} k[x_1, ..., x_n]_{m_a} \subset k(x_1, ..., x_n)$$

i.e. it consists of the rational functions that are regular at all points of U.

Examples. (a) For $g \in k[x_1, ..., x_n]$, let $U_g := k^n - X(g)$. The "other" localization:

$$k[x_1, ..., x_n]_g = \left\{ \frac{f}{g^m} \mid f \in k[x_1, ..., x_n] \text{ and } m \geq 0 \right\} / \sim$$

at the multiplicative set $\{1, g, g^2, ...\}$ is clearly contained in the ring $\mathcal{O}_{\mathbb{A}_k^n}(U_g)$. We will see (Proposition 2.7) that $k[x_1, ..., x_n]_g = \mathcal{O}_{\mathbb{A}_k^n}(U_g)$. In particular, this is a strictly larger ring than $k[x_1, ..., x_n]$.

(b) Let $U = k^2 - \{(0, 0)\}$. Then $\mathcal{O}_{\mathbb{A}_k^2}(U) = k[x_1, x_2]$. This is equivalent to:

$(*)$ if $g \in k[x_1, x_2]$ and $g(0, 0) = 0$, then $g(a) = 0$ for some $a \neq (0, 0)$

Intuitively, the loci $X(g) \subset k^2$ are irreducible algebraic curves and not points (see Corollary 2.14.) On the other hand, $X(x_1^2 + x_2^2) = (0, 0) \in \mathbb{R}^2$, so when we prove this, we will clearly need the assumption $k = \overline{k}$.

Affine Space $\mathbb{A}_k^n$ is, by definition, the vector space k^n with the Zariski topology and sheaf of regular functions (assuming, as always, that k is algebraically closed).

Affine Varieties. We now generalize from $\mathbb{A}_k^n$ to **affine varieties**.

Let $P \subset k[x_1, ..., x_n]$ be a prime ideal, $X = X(P) \subset k^n$ and $k[X] := k[x_1, ..., x_n]/P$.

With the assumption that $k = \overline{k}$, the points $x \in X$ correspond to maximal ideals $m_x \subset k[X]$ via the Hilbert Nullstellensatz and the ideal correspondence:

$$\{\text{(maximal) ideals in } k[X]\} \leftrightarrow \{\text{(maximal) ideals in } k[x_1, ..., x_n] \text{ containing } P\}$$

Notice that the elements of $k[X]$ are functions $f : X \to k$, and

$$f(x) = g(x) \text{ for all } x \in X \text{ if and only if } f = g \in k[X]$$

since $f(x) = g(x)$ for all x if and only if $f - g \in m_x \subset k[X]$, and $0 = \bigcap_{x \in X} m_x$, which is the well-known fact that a prime ideal P is the intersection of all maximal ideals that contain P. We now get an intrinsic definiton of the Zariski topology:

$$Z \subset X \text{ is closed if and only if } Z = X(I) \text{ for some } I \subset k[X]$$

which agrees with the topology induced from k^n (by the ideal correspondence).

Let $k(X)$ be the fraction field of $k[X]$. These are the **rational functions** on X.

Definition 2.5. A rational function $\phi \in k(X)$ is **regular** at $x \in X$ if $\phi \in k[X]_{m_x}$.

Remark. This is the same as Definition 2.3, but in this case one needs to take some care since unlike the ring $k[x_1, ..., x_n]$, the coordinate rings $k[X]$ are often **not** UFDs. In that case, there are many choices for f and g so that $\phi = f/g$. To prove that ϕ is regular at x, we just need to find **one** such expression with $g(x) \neq 0$.

Example. Consider the ideal $P = \langle x_1x_4 - x_2x_3 \rangle \subset k[x_1, x_2, x_3.x_4]$. This is prime, but evidently $k[X]$ is not a UFD since by construction:

$$x_1x_4 = x_2x_3$$

and x_i are irreducible elements of $k[X]$. Thus:

$$\phi = \frac{x_1}{x_2} = \frac{x_3}{x_4} \in k(X)$$

and ϕ is regular away from the intersection $X(x_2) \cap X(x_4) = \{(s, 0, t, 0)\}$ whereas $X(x_2) = \{(s, 0, t, 0)\} \cup \{(0, 0, t, s)\}$ and $X(x_4) = \{(s, 0, t, 0)\} \cup \{(s, t, 0, 0)\}$, which shows that no single form for ϕ accounts for all the regular points.

Definition 2.6. The **sheaf of regular functions** $\mathcal{O}_X$ on the affine variety X is:

$$\mathcal{O}_X(U) = \bigcap_{x \in U} k[X]_{m_x} \text{ (the rational functions that are regular at all points of } U\text{)}$$

Proposition 2.7. If $g \in k[X]$, let $U_g = X - X(g) \subset X$, Then:

$$\mathcal{O}_X(U_g) = k[X]_g$$

In particular, $\mathcal{O}_X(X) = k[X]$.

Proof. Suppose $\phi \in \mathcal{O}_X(U_g)$ and consider the *ideal of denominators* of ϕ:

$$I = \{h \in k[X] \mid h\phi \in k[X]\}$$

Then $X(I) \subset X$ is the set of points where ϕ is **not** regular, and so $X(I) \subset X(g)$, by definition of $\mathcal{O}_X(U_g)$. Thus $g \in I(X(I)) = \text{rad}(I)$ by the Nullstellesatz. Thus:

$$g^n\phi \in k[X] \text{ for some } n$$

i.e. $\phi \in k[X]_g$. This gives $\mathcal{O}_X(U_g) \subset k[X]_g$. The other inclusion is immediate. □

Remark. The open subsets $U_g \subseteq X$ are a *basis* for the Zariski topology on X. Indeed, if $X(I) \subset X$ is closed then $I = \langle g_1, .., g_r \rangle$ for finitely many $g_1, ..., g_r \in k[X]$ (since $k[X]$ is Noetherian) and then:

$$U = X - X(I) = X - (X(g_1) \cap \cdots \cap X(g_r)) = U_{g_1} \cup \cdots U_{g_r}$$

so in fact every open set is the union of *finitely many* basic open sets U_g.

Definition 2.8. The *stalk* at $x \in X$ of a presheaf $\mathcal{A}$ of commutative rings is the direct limit over the open neighborhoods $x \in U$:

$$\mathcal{A}_x = \lim_{\rightarrow} \mathcal{A}(U) \text{ (via the restriction maps } \rho_{U,V}\text{)}$$

Recall that the direct limit is a commutative ring with ring homomorphisms:

$$\mathcal{A}(U) \to \mathcal{A}_x$$

that are compatible, in the sense that they factor through $\rho_{U,V}$ whenever $V \subseteq U$ (and a universal property).

Examples. (a) The stalks of the constant sheaf $\mathcal{A}(U) = A$ are $\mathcal{A}_x = A$.

(b) The stalks of the sheaf $\mathcal{C}^\infty_M$ of differentiable functions on a differentiable manifold M are the **germs** of $\mathcal{C}^\infty$ functions at $x \in M$. These are in fact all isomorphic rings, since every point $x \in M$ has an open neighborhood $x \in B$ diffeomorphic to the unit open ball in $\mathbb{R}^n$, and the direct limit doesn't change when it is restricted to open neighborhoods of x that are contained in B.

(c) The stalks $\mathcal{O}_{X,x}$ of the sheaf $\mathcal{O}_X$ of regular functions on X are the local rings:

$$\mathcal{O}_{X,x} = k[X]_{m_x}$$

with the inclusion maps $\mathcal{O}_X(U) \subset k[X]_{m_x}$ for each neighborhood of x. Notice that when $x \in U_g$, then this corresponds to $k[X]_g \subset k[X]_{m_x}$

Taking Stock. An affine variety (over $k = \overline{k}$) converts an **algebraic object:**

$$A = \text{a finitely generated } k\text{-algebra integral domain}$$

into a **geometric object** (independent of the choice of generators $x_1, ..., x_n \in A$). The geometric object consists of:

- Points. $X = \{\text{maximal ideals of } A\}$
- Topology. $U = X - X(I)$ are the open sets of the Zariski topology, with a basis of open sets given by $U_g = X - X(g)$ and with respect to which X is irreducible.
- Sheaf. The sheaf of regular functions $\mathcal{O}_X$ with $\mathcal{O}_X(U_g) = A_g$ and $\mathcal{O}_{X,x} = A_{m_x}$.

Following the scheme literature, we will denote this as a pair:

$$(X, \mathcal{O}_X) = \text{maxspec}(A)$$

and call it the "spectrum" of maximal ideals in A. To interpret maxspec as a *functor*, we need affine varieties to belong to a *category*. The category we use will be the category of sheaved spaces:

$$(X, \mathcal{O}_X)$$

consisting of a topological space X and a *sheaf of functions* $\phi : U \to k$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two sheaved spaces.

Definition 2.9. A *continuous* map $\eta : X \to Y$ is a morphism of sheaved spaces if and only if the pull-back of functions defined by $\mathcal{O}_Y$ are functions defined by $\mathcal{O}_X$, i.e.

$$\eta^*(\mathcal{O}_Y(U)) \subset \mathcal{O}_X(f^{-1}(U))$$

for all open subsets $U \subset Y$, where the pull-back is defined by $\eta^*(f) := f \circ \eta$.

Quasi-Affine and Affine Varieties in the Category of Sheaved Spaces. Let $W \subset X = \text{maxspec}(A)$ be an open subset and define the sheaf of functions $\mathcal{O}_W$ by:

$$\mathcal{O}_W(U) = \mathcal{O}_X(U)$$

Any sheaved space isomorphic to some $(W, \mathcal{O}_W)$ is a **quasi-affine** variety in the category of sheaved spaces. Sheaved spaces that are isomorphic to one of the sheaved spaces $\text{maxspec}(A) = (X, \mathcal{O}_X)$ are **affine** varieties.

Proposition 2.10. Each of the quasi-affine varieties $(U_g, \mathcal{O}_{U_g})$ for $g \in A$ is affine.

Proof. Given A and $g \in A$, consider the localization A_g.

(1) A_g is a finitely generated k-algebra integral domain. If $x_1, ..., x_n \in A$ are generators of A, then $A_g \subset k(A)$ is generated by $x_1, ..., x_n, x_{n+1} = 1/g$.

(2) There is a natural bijection of points:

$$i : \text{maxspec}(A_g) \to U_g \subset \text{maxspec}(A)$$

given by the localization of maximal ideals:

$$\{\text{maximal ideals in } A_g\} \xrightarrow{i} \{\text{maximal ideals in } A \text{ not containing } g\}$$

(3) This induces a bijection of open sets (forming a basis of the two topologies):

$$\{U_\phi \subset \text{ maxspec}(A_g)\} \xrightarrow{i} \{U_{\phi g^n} \subset U_g \subset A\} \text{ (for all large enough } n)$$

so the bijection i is a homeomorphism of the Zariski topologies, and

(4) The pull-back via i of regular functions (on the basis of open sets) identifies:

$$i^* : A_{\phi \cdot g^n} = (A_g)_\phi \subset k(A)$$

as the same subrings of $k(A)$.

Example. (a) If $A = k[x]$ and $g = x$, then the quasi-affine varieties:

$$\mathbb{A}^1_k - \{0\} \text{ and the hyperbola } X(x_1x_2 - 1) \subset \mathbb{A}^2_k$$

are isomorphic as sheaved spaces, via the projection map. Hence $\mathbb{A}^1_k - \{0\}$ is affine.

(b) The quasi-affine variety $(W = \mathbb{A}^2_k - \{(0, 0)\}, \mathcal{O}_W)$ is not affine.

This will be a consequence of the (still unproven) fact that $\mathcal{O}_{\mathbb{A}^2_k}(W) = k[x_1, x_2]$ since $X = \{\text{maximal ideals in } \mathcal{O}_X(X)\}$ for affine varieties and W lacks the origin.

Definition 2.11. maxspec is a contravariant functor, with the assignment:

$$\text{maxspec}(\alpha : A \to B) = (\alpha^* : \text{maxspec}(B) \to \text{maxspec}(A))$$

for all homomorphisms of commutative rings, defined by:

• $\alpha^*(m_b) = \alpha^{-1}(m_b)$ from which we conclude that $(\alpha^*)^{-1}(U_g) = U_{\alpha(g)}$ for $g \in A$. Since these open sets form a basis of the Zariski topology on $Y = \text{maxspec}(A)$, it follows that α^* is a continuous map of topological spaces.

• $(\alpha^*)^*(A_g) = B_{\alpha(g)}$ from which we conclude that α^* pulls regular functions on U_g back to regular functions on $U_{\alpha(g)}$. Since these are a basis for the topology, it follows again that α is a morphism in the category of sheaved spaces,

Note. As defined, maxspec is an **equivalence** of categories from finitely generated commutative k-algebra domains to the full subcategory of affine varieties within the category of sheaved spaces.

Examples. (a) If $B = k[x_1, ..., x_n]/P$, then the surjective ring homomorphism:

$$\alpha : k[x_1, ..., x_n] \to B$$

corresponds to the **closed embedding** of $X = \text{maxspec}(B)$ in the affine space $\mathbb{A}^n_k$, identifying X with the affine variety $X(P)$.

(b) If $g \in A$ and $B = A_g$, then the injective ring homomorphism $\alpha : A \to A_g$ corresponds to the **open embedding** $U_g \subset \text{maxspec}(A)$. But in general, injective ring homomorphisms do **not** correspond to open embeddings. For example:

$$\alpha : k[x_1] \to k[x_1, x_2] \text{ corresponds to the } \textbf{projection } p : \mathbb{A}^2_k \to \mathbb{A}^1_k$$

(c) Suppose $A = k[x_1, ..., x_n]/P$ and $B = k[y_1, ..., y_m]/Q$ and:

$$\alpha : A \to B \text{ is a commutative ring homomorphism}$$

Then by lifting the images $\alpha(x_i) = \overline{f}_i$ to polynomials $f_i(y_1, ..., y_m) \in k[y_1, ..., y_m]$, we get a commuting diagram of ring homomorphisms:

$$\begin{array}{ccc} k[x_1, ..., x_n] & \stackrel{\widetilde{\alpha}}{\to} & k[y_1, ..., y_m] \\ \downarrow & & \downarrow \\ A & \stackrel{\alpha}{\to} & B \end{array}$$

from which we conclude:

($*$) Every morphism $\eta : X(Q) \to X(P)$ of embedded affine varieties $X(Q) \subset \mathbb{A}^m_k$ and $X(P) \subset \mathbb{A}^n_k$ *lifts* to a morphism of affine spaces:

$$\widetilde{\eta}(y_1, ..., y_m) = (f_1(y_1, ..., y_m), ..., f_n(y_1, ..., y_m))$$

given by polynomial functions of the coordinates. In that sense, morphisms of affine varieties are polynomial (regular) maps.

The functor maxspec allows us to define geometric properties of morphisms of affine varieties (sheaved spaces) in terms of algebraic properties of homomorphisms of commutative rings. This will be a recurring theme. We start with:

Definition 2.12. A morphism $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of affine varieties is **finite onto its image** if the corresponding ring homomorphism:

$$\alpha = \phi^* : \mathcal{O}_Y(Y) = A \to \mathcal{O}_X(X) = B$$

makes B into a finite A-module.

Proposition 2.12. If a morphism ϕ of affine varieties is finite onto its image, then:

(a) $\phi : X \to Y$ is finite-to-one.

(b) $\phi : X \to Y$ takes (irreducible) closed sets to (irreducible) closed sets.

(c) If in addition, $\alpha = \phi^*$ is injective, then $\phi : X \to Y$ is **surjective**.

Corollary 2.13. If A is a finitely generated commutative algebra domain over k and $k(A)$ has transcendence degree d over k, then there is an inclusion:

$$\alpha : k[y_1, ..., y_d] \subset A \text{ such that } A \text{ is a finite } k[y_1, ..., y_d]\text{-module}$$

by Noether Normalization, and so the morphism of varieties:

$$\alpha^* : \text{maxspec}(A) \to \mathbb{A}^d_k$$

is surjective and finite-to-one by Proposition 2.12. If $x_1, ..., x_n$ are generators of A (as a k-algebra) then the y_i may be chosen so that the finite-to-one morphism α^* lifts (in the sense of ($*$) above) to a linear **projection** $\mathbb{A}^n_k \to \mathbb{A}^d_k$.

Corollary 2.14. If $g \in k[x_1, ..., x_n]$ for $n > 1$, then $X(g)$ is not a single point.

Proof. A single point does not project onto $\mathbb{A}^d_k$ when $d > 0$.

Assignment 2.

1. (a) Explain why the field $k(A)$ of rational functions on A is:

$$\lim_{\rightarrow} \mathcal{O}_X(U) \text{ taken over the direct system of all open subsets of } X$$

If $P \subset A$ is prime, express $k[A]_P$ as a direct limit over a system of open sets in X. The **dimension** of X is the transcendence degree of the field extension $k \subset k(A)$.

(b) Show that the rings $k[A]_P$ are not finitely generated k-algebras when $d > 0$.

2. The **Zariski cotangent space** of X =maxspec(A) at a point $x \in X$ is:

$$m_x/m_x^2$$

where $m_x \subset A$ is the maximal ideal corresponding to x.

(a) Identify $m_x/m_x^2 = m/m^2$ where m is the (unique) maximal ideal in $\mathcal{O}_{X,x}$.

(b) If $A = k[x_1, ..., x_n]/\langle f_1, ..., f_m\rangle$, show that the Zariski cotangent space to X at $a \in X \subset \mathbb{A}_k^n$ is the intersection of the affine linear subspaces:

$$\sum \frac{\partial f_i}{\partial x_j} x_j = \sum \frac{\partial f_i}{\partial x_j} a_i; \ i = 1, ..., m$$

through $a = (a_1, ..., a_n) \in k^n$ (which becomes the origin of the vector space m_x/m_x^2).

3. Discuss the points (1)-(4) in the proof of Proposition 2.10.

4. The presheaves of abelian groups (or commutative rings with 1) on a topological space X form a **category**, in which the morphisms $f : \mathcal{A} \to \mathcal{B}$ of presheaves are natural transformations of functors, i.e. collections of homomorphisms:

$$f_U : \mathcal{A}(U) \to \mathcal{B}(U) \text{ with } \rho^{\mathcal{B}}_{U,V} \circ f_U = f_V \circ \rho^{\mathcal{A}}_{U,V}$$

In particular, a morphism of presheaves determines homomorphisms of the **stalks**

$$f_x : \mathcal{A}_x \to \mathcal{B}_x \text{ for each } x \in X$$

Let A be a fixed abelian group and let $\mathcal{A}^+$ be the presheaf of continuous functions:

$$\alpha : U \to A \text{ (for the discrete topology on } A)$$

(i) Show that $\mathcal{A}^+$ is a sheaf of abelian groups with stalks $\mathcal{A}^+_x = A$.

(ii) Show that the natural morphism from the constant presheaf to $\mathcal{A}^+$:

$$f : \mathcal{A} \to \mathcal{A}^+ \text{ (regarding a constant function as a continuous function)}$$

induces an isomorphism on stalks.

Remark. This is an example of the *sheafification* of a presheaf.

5. If $(X, \mathcal{O}_X)$ is a sheaved space, we saw that an open subset $W \subset X$ is a sheaved space with the induced topology and sheaf $\mathcal{O}_W(U) = \mathcal{O}_X(U)$.

More painfully, if $Z \subset X$ is a closed set with the induced topology, then:

$$\mathcal{O}_X|_Z(U) = \{f : U \to k \mid \exists g \in \mathcal{O}_X(W) \text{ with } W \cap Z = U \text{ and } g|_U = f\}$$

defines a presheaf of rings that may or may not itself be a sheaf. When $(X, \mathcal{O}_X) =$ maxspec(A) and $Z = X(P)$ for a prime ideal $P \subset A$, convince yourself that maxspec$(A/P) = (Z, \mathcal{O}_Z)$ sheafifies the presheaf $\mathcal{O}_X|_Z$. It is in this sense that we say that a closed embedding is an isomorphism from a sheaved space to a closed subset of a sheaved space.

6. (a) Prove that $\alpha : A \to B$ is **injective** if and only if the image of

$$\alpha^* : Y = \text{maxspec}(B) \to X = \text{maxspec}(A)$$

is **dense** in X. Such a morphism of affine varieties is called **dominant**.

(b) Use (a) to factor an arbitrary morphism $\phi : Y \to X$ as:

$$Y \stackrel{\phi}{\to} Z \subset X$$

a dominant morphism followed by a closed embedding.

(c) Consider the injective ring homomorphism:

$$\alpha : k[x_1, x_2] \to k[y_1, y_2];\ \text{given by } y_1 = x_1, y_2 = x_1x_2$$

Describe the resulting morphism of affine planes:

$$\alpha^* : \mathbb{A}^2_k \to \mathbb{A}^2_k$$

and conclude that the image of α^*, while dense, is neither open nor closed.

7. Determine the commutative algebra facts you need to prove Proposition 2.12. Then look up their proofs.