1. (a) Find the general solution $u(x, t)$ of the following equation
   
   $u_t + tu_x = 0 \quad (-\infty < x < \infty, \ t > 0)$.

   (b) Find the solution of this equation satisfying the initial condition
   
   $u(x, 0) = \begin{cases} 
   4 & \text{if } |x| < 2, \\
   0 & \text{otherwise}
   \end{cases}$

   (c) Sketch the solution for three values $t = 0, 1, 2$.

2. Find the frequencies of sound produced by an organ pipe (of length $L$) open at both ends. The dynamics of air in the pipe is described by the boundary value problem

   $u_{tt} = c^2 u_{xx}$,

   $u(0, t) = 0, \quad u(L, t) = 0$

   for the pressure variation $u = p - p_0$; $p = p(x, t)$ is the pressure in the pipe, $p_0$ is the atmospheric pressure.

   What is the fundamental mode?

   Will its sound become higher or lower if the pipe is played in water (instead of air)?

   [For reference, the values of sound speed are: in air $340 m/s$, in water $1500 m/s$]

3. Solve the boundary value problem

   $u_{tt} = c^2 u_{xx}$,

   $u(0, t) = u(L, t) = 0$,

   $u(x, 0) = \sin \frac{\pi}{L} x, \quad u_t(x, 0) = \frac{1}{10} \sin 3 \frac{\pi}{L} x$.

   [No integration is needed to solve this problem.]

4. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the entire line $-\infty < x < +\infty$, with initial conditions

   $u(x, 0) = \begin{cases} 
   4 & \text{if } |x| < 2, \\
   0 & \text{otherwise}
   \end{cases} \quad u_t(x, 0) = 0$

   Taking $c = 10$, sketch $u(x, t)$ for three values $t = 0, 1, 2$.

5. Consider cooling of a potato. Its temperature $u(x, y, z, t)$ obeys the following boundary value problem

   $u_t = c^2 \left( u_{xx} + u_{yy} + u_{zz} \right) \quad (x^2 + y^2 + z^2 < a^2)$,

   $u(x, y, z, t) = 0 \quad \text{if } x^2 + y^2 + z^2 = a^2$,

   $u(x, y, z, 0) = M \ (= 100^\circ C) \quad \text{if } x^2 + y^2 + z^2 < a^2$,

   $a$ is the radius of the potato.

   (a) Find the temperature $u(x, t)$ of the potato for $t > 0$.

   (b) Sketch a few snapshots of the temperature as a function of $x$ [at different fixed values of time].

   (c) Estimate time $t_0$ when the maximal temperature in the potato is $50^\circ$.
6. Consider heat propagation in a completely insulated bar (its lateral sides and both ends are insulated). The temperature dynamics is described by the following boundary value problem

\[ u_t = c^2 u_{xx}, \]
\[ u_x(0, t) = 0, \quad u_x(L, t) = 0. \]

Derive from these equations that the quantity

\[ Q = \int_0^L u(x, t) \, dx \]

(which is proportional to the total heat energy inside the bar) is conserved.

7. Solve the initial value problem

\[ u_{tt} + u_{xxxx} = 0, \quad u(x, 0) = e^{-\frac{ax^2}{2}}, \quad u_t(x, 0) = 0 \quad (-\infty < x < \infty); \]

\[ a \] is a positive parameter.

8. The solution \( u(x, t) \) of the initial value problem for the heat equation

\[ u_t = c^2 u_{xx}, \quad u(x, 0) = f(x) \quad (-\infty < x < \infty, t > 0) \]

is given by the integral

\[ u(x, t) = \int_{-\infty}^{\infty} G(x - s, t) f(s) \, ds \quad \text{with} \quad G(x, t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{x^2}{4c^2 t}} \]

(Section 7.4, Theorem 1, Formula (4), Page 421).

(a) Show that the solution of the boundary value problem

\[ u_t = c^2 u_{xx} \quad (x > 0, t > 0), \quad u(0, t) = 0 \quad \text{(fixed zero 1 temperature at the end)}, \quad u(x, 0) = f(x) \quad \text{(initial temperature)} \]

can be found by the integral

\[ u(x, t) = \int_0^\infty [G(x - s, t) - G(x + s, t)] f(s) \, ds \]

(b) Show that the solution of the boundary value problem

\[ u_t = c^2 u_{xx} \quad (x > 0, t > 0), \quad u_x(0, t) = 0 \quad \text{(insulated end)}, \quad u(x, 0) = f(x) \quad \text{(initial temperature)} \]

can be found by the integral

\[ u(x, t) = \int_0^\infty [G(x - s, t) + G(x + s, t)] f(s) \, ds \]

[Hint: Use the method of images]

9. Estimate the age of the earth [under the assumption that there are no heat sources inside the earth and no convection]. Assume that

- initially the earth has temperature of the melted rock, \( M = 2000^\circ C \);
- \( c^2 = 1.2 \times 10^{-6} \frac{m^2}{s} \);
- the temperature of the surrounding atmosphere has always been \( 0^\circ C \);
- current value of geothermal gradient is \( \alpha = 0.037 \frac{^\circ C}{m} \).