The goal is Problem 3, but its solution is based on the previous problems.

1. **Sturm-Liouville Operator.**

   (a) Consider an operator
   
   \[ L : \phi(x) \mapsto -\frac{1}{\mu(x)} \frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + q(x)\phi \]
   
   acting in the space \( V \) of functions \( \phi(x), x \in (a, b) \), such that \( \phi(a) = \phi(b) = 0 \). \( a, b \) are real numbers; \( \mu(x), p(x), q(x) \) are given real-valued functions, and \( \mu(x) > 0 \).
   
   Define a dot (inner) product \( \langle \cdot, \cdot \rangle \) on \( V \) so that the operator \( L \) is symmetric:
   
   \[ \langle L\phi, \psi \rangle = \langle \phi, L\psi \rangle, \quad \forall \phi, \psi \in V \]
   
   (similar to symmetric matrices).
   
   [You need to use a “weight” function; remember to show that your definition is indeed dot product, satisfying the four axioms.]

   (b) Now consider the same operator \( L \) acting in a different space
   
   \[ W = \{ \phi(x), x \in (a, b) : \alpha_1 \frac{\partial \phi}{\partial x} + \alpha_2 \phi = 0 \text{ when } x = a, \quad \beta_1 \frac{\partial \phi}{\partial x} + \beta_2 \phi = 0 \text{ when } x = b. \} \]
   
   \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are real numbers; \( V \) is a particular case of \( W \).
   
   Again, define a dot (inner) product \( \langle \cdot, \cdot \rangle \) on \( W \) so that the operator \( L \) is symmetric.
   
   (Show that \( L \) is indeed symmetric with your definition.)

2. (a) Show that all eigenvalues of \( L \) (from the Problem #1) are real.
   
   (b) Show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal.
   
   (c) Argue that the eigenfunctions form a “complete” set, meaning that any (“usual”) function can be represented as a superposition of eigenfunctions.

3. Consider heat conduction in a rod with a linear heat source/sink term and variable thermal properties:
   
   \[ cp \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ K_0 \frac{\partial u}{\partial x} \right] + su, \]
   
   where the specific heat \( c(x) \), the density \( \rho(x) \), the thermal conductivity \( K_0(x) \), and the source coefficient \( s(x) \) are given functions, \( a < x < b \). Consider general homogeneous boundary conditions:
   
   \[ \alpha_1 \frac{\partial u}{\partial x} + \alpha_2 u = 0 \quad \text{at the left end } x = a, \]
   
   \[ \beta_1 \frac{\partial u}{\partial x} + \beta_2 u = 0 \quad \text{at the right end } x = b. \]
   
   Assuming the initial condition \( u(x, 0) = f(x) \), find the temperature \( u(x, t) \) for later time \( t > 0 \).
   
   [Your solution can involve several steps:
   
   (a) The method of separation of variables to find many product solutions \( u(x, t) = \phi(x)G(t) \), satisfying the equation and the boundary conditions. This includes derivation of the eigenvalue problem for the function \( \phi(x) \).
   
   (b) Argue why you can apply the superposition principle and obtain huge family of solutions.
   
   (c) Satisfying the initial conditions. Refer to the previous problems #1 and #2 for two things: You need to argue that an arbitrary initial condition \( f(x) \) can be represented as a superposition of the eigenfunctions; and you need to show how to find the coefficients in this superposition.]