Problem 1. Least square solution.
Consider a linear system
\[ Ax = b \]
with \( m \) equations and \( n \) unknowns (\( A \) is a \( m \times n \)-matrix, \( b \) is a \( m \)-vector, and \( x \) is a \( n \)-vector). If \( m > n \), this system is over-determined and is likely to have no exact solution. So, we look for the best approximate solution \( x_0 \), that minimizes the distance
\[ ||Ax - b||. \]
1. Show that \( x_0 \) is obtained as solution of the linear system \( A^T A x_0 = A^T b \).
   \text{Suggestion:} \text{Consider } x = x_0 + z \text{ and show that}
   \begin{align*}
   ||Ax - b||^2 &= [A(x_0 + z) - b]^T [A(x_0 + z) - b] \\
   &= [Ax_0 - b]^T [Ax_0 - b] + [Az]^T [Az]
   
   \end{align*}
2. What is the size of matrix \( A^T A \)?
3. Show that \( A^T A \) is a symmetric matrix.
4. Show that \( A^T A \) is a semi-positive-definite matrix.
5. Is \( A^T A \) always positive-definite?

Problem 3. Let \( A \) be an \( n \times n \) matrix. Prove:
A set of its eigenvectors \( v_1, v_2, \ldots, v_k \) (\( k \leq n \)) — corresponding to distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \) — is linearly independent.

Problem 4.
1. Prove: If an \( n \times n \) matrix \( A \) has full set of \( n \) real eigenvectors, then it is similar to the diagonal matrix (i.e. there is a non-singular matrix \( S \) and there is a diagonal matrix \( \Lambda \) such that \( A = SAS^{-1} \)).
2. What is \( S \)? What is \( \Lambda \)?
3. Give an example of a matrix that does not have a full set of real eigenvectors, but has a full set of complex eigenvectors.
4. Give an example of a matrix that does not have a full set of complex eigenvectors.