Quiz Scores (out of 10): n = 140; mean = 5.9
25th percentile = 3.5; median (50th percentile) = 6.0; 75th percentile = 8.5

1. (5 pts) Find $\frac{dy}{dx}$ in each of the following:

i. (2 pts) $y = (x^2 + 1)^{\sin x}$.

Using logarithmic differentiation we have:

$$\ln y = \ln[(x^2 + 1)^{\sin x}] = (\sin x)\ln(x^2 + 1).$$

Then differentiating implicitly with respect to $x$, we find:

$$\frac{1}{y} \frac{dy}{dx} = \sin x \left( \frac{2x}{x^2 + 1} \right) + (\cos x)\ln(x^2 + 1).$$

Solving for $\frac{dy}{dx}$ and replacing $y$ by $(x^2 + 1)^{\sin x}$ we have:

$$\frac{dy}{dx} = (x^2 + 1)^{\sin x} \left[ \sin x \left( \frac{2x}{x^2 + 1} \right) + (\cos x)\ln(x^2 + 1) \right].$$

**Comment:** Logarithmic differentiation, and implicit differentiation are very useful tools!

ii. (3 pts) $y = \sin^{-1}(e^{2x})$

Using the Chain Rule we have:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(e^{2x})^2}} \frac{d(e^{2x})}{dx} = \frac{2e^{2x}}{\sqrt{1-(e^{4x})}}$$

**Comment:** In differentiating $f(x) = e^{2x}$ we need the Chain Rule. $\frac{d(e^{2x})}{dx} = 2e^{2x}$. Also note that you need to memorize the derivatives of $y = \sin^{-1}(x), y = \cos^{-1}(x)$, and $y = \tan^{-1}(x)$. 
2. (3 pts) Evaluate the following definite integral.

\[
\int_{0}^{\pi/2} \frac{\sin x}{1 + \cos^2 x} \, dx
\]

If we let \( u = \cos x \), then \( du = -\sin x \, dx \). Note that \( u = 1 \), when \( x = 0 \), and \( u = 0 \), when \( x = \pi/2 \). The above integral becomes:

\[
\int_{0}^{\pi/2} \frac{\sin x}{1 + \cos^2 x} \, dx = - \int_{1}^{0} \frac{1}{1 + u^2} \, du = -\arctan u \Big|_{1}^{0} = -\arctan(0) + \arctan(1)
\]

Since \( \arctan(0) = 0 \), \( \arctan(1) = \pi/4 \), the above integral equals \( \pi/4 \).

**Comment:** If you let \( u = 1 + \cos^2 x \), then \( du = 2\cos x \sin x \, dx \).

3. (2 pts) Suppose that \( \frac{dy}{dx} + 2y = 3 \) and that \( y = 1 \) when \( x = 0 \). Find \( y \) as a function of \( x \).

The above equation is of the form, \( \frac{dy}{dx} + P(x)y = Q(x) \) with \( P(x) = 2 \). Since \( \int P(x) \, dx = \int 2 \, dx = 2x \), the integrating factor is \( e^{2x} \). Multiplying the differential equation on both sides by this integrating factor we get:

\[
\frac{dy}{dx} e^{2x} + 2ye^{2x} = 3e^{2x}, \text{ or } \frac{d(ye^{2x})}{dx} = 3e^{2x}.
\]

Integrating both sides of the equation we have: \( ye^{2x} = \int 3e^{2x} \, dx = (3/2)e^{2x} + C \), \( C \) a constant. If we replace \( y \) by 1, and \( x \) by 0, then we get \( C = -1/2 \) and solving for \( y \) we have \( ye^{2x} = (3/2) - (1/2)e^{-2x} \).

**Comment:** Note that \( \int 3e^{2x} \, dx = \frac{3}{2}e^{2x} \). Remember that you can check your answer quickly in a small integral like this by simply differentiating. If you had \( \int 3e^{2x} \, dx = 6e^{2x} \), then differentiating, you’d get \( \frac{d(6e^{2x})}{dx} = 6[2e^{2x}] = 12e^{2x} \) and you know that you were off by a constant.