MATH 1260 - Test # 2 Sample Questions

Formula Sheet

- The dot product of two plane vectors \( \vec{u} = \langle u_1, u_2 \rangle \) and \( \vec{v} = \langle v_1, v_2 \rangle \) is
  \[
  \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.
  \]

- The dot product of two space vectors \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) is
  \[
  \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.
  \]

- The cross product of two vectors \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) is
  \[
  \vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle.
  \]

- The projection of a vector \( \vec{u} \) onto a vector \( \vec{v} \) is
  \[
  \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \cdot \vec{v}.
  \]

- The determinant of a 2 by 2 matrix is
  \[
  \text{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.
  \]

- The determinant of a 3 by 3 matrix is
  \[
  \text{det} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c}).
  \]

- The gradient of a function \( f(x, y) \) is
  \[
  \nabla f = \langle f_x, f_y \rangle.
  \]

- The gradient of a function \( f(x, y, z) \) is
  \[
  \nabla f = \langle f_x, f_y, f_z \rangle.
  \]

- The tangent plane of a function \( f(x, y) \) at a point \( P = (a, b) \) is
  \[
  z = f_x(P) \cdot (x - a) + f_y(P) \cdot (y - b) + f(P).
  \]

- Chain Rule
  \[
  D(g \circ f)_P = Dg_{f(P)} \cdot Df_P.
  \]
(1) Calculate the following limits or show that they do not exist:

(a) \( \lim_{(x,y) \to (0,0)} \frac{2xy^2}{x^2+y^2} \).

(b) \( \lim_{(x,y) \to (0,0)} \frac{x^3+y^4}{x+2y^2} \).

(c) \( \lim_{(x,y) \to (0,0)} \frac{x+2y^2}{3x^2+2y^2} \).

(d) \( \lim_{(x,y) \to (0,0)} \frac{x^6+y^4}{x^2y} \).

(e) \( \lim_{(x,y) \to (0,0)} \frac{x^2y}{3x^2+2y^2} \).

(f) \( \lim_{(x,y) \to (0,0)} \frac{x^4+3y^2}{3x^3+2\sqrt{y}} \).

(g) \( \lim_{(x,y) \to (0,0)} \frac{x^2+y^2}{-3x^3-y^2} \).

(h) \( \lim_{(x,y) \to (0,0)} \frac{x^2y^5}{3x^3+2y^2} \).

(i) \( \lim_{(x,y) \to (0,0)} \frac{2x^4+3y^{10}}{x^2y^2} \).

(j) \( \lim_{(x,y) \to (0,0)} \frac{2x^2+3y^4}{2x^2+3y^4} \).

(2) Find a function \( f: \mathbb{R}^3 \to \mathbb{R}^2 \) such that \( f(\vec{v}) = \vec{0} \) for every vector \( \vec{v} \) in the plane of equation \( 2x - 3y + z = 0 \).

(3) Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be the function \( f(x, y) = 3x - 2y \). Show that
   (a) for every number \( c \), the set of vectors in \( \mathbb{R}^2 \) mapping to \( c \) is a line;
   (b) all the lines in part (a) are parallel.

(4) Let \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( f(x, y) = (2x - y, -x + 2y) \). Find all vectors \( \vec{v} \) such that \( f(\vec{v}) = c\vec{v} \) for some number \( c \).

(5) Let \( A \) be a 2 by 2 matrix. Show that, if there exists a non-zero vector \( \vec{a} \) such that \( \vec{a} \cdot A = \vec{0} \), then \( \det(A) = 0 \).

(6) Find all the numbers \( a \) and all non-zero vectors \( \vec{v} \) such that

\[
\vec{v} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = a\vec{v}.
\]

(7) Let \( f(x, y) = 2x^2y^3 - 3xe^y + x\ln(x) - \sin(xy) \).
   (a) Find the partial derivatives \( f_x \) and \( f_y \).
   (b) Find the gradient of \( f \) at the point \( (1, 0) \).

(8) Find \( g_x \), \( g_y \), and \( g_z \) for the function \( g(x, y, z) = (x + z^2)\cos(y)\sin(2y) \).
(9) Show that, if $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function and $P$ is a point, then the gradient $\nabla f(P)$ of $f$ at $P$ is equal to $\langle Df_{(1,0)}(P), Df_{(0,1)}(P) \rangle$. (Remember that $Df_{\vec{u}}(P)$ denotes the directional derivative of $f$ at the point $P$ in the direction $\vec{u}$)

(10) Consider the graph of the function $f(x, y) = 2x^2 \ln(xy) + 3\sqrt{y}$. Find the tangent plane at the point $(1, 1)$.

(11) Consider the functions

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad f(x_1, x_2, x_3) = \left( x_1 x_3, 3x_1 + \frac{x_2}{x_3} \right)$$

$$g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad g(y_1, y_2) = \left( \sqrt{y_1}, 3y_1^2 - 2y_1 e^{y_1 y_2} \right).$$

Find the derivative of $g \circ f$ at the point $(1, -3, 1)$.

(12) Let $f(x, y, z) = \left( x^2 + y^2 + z^2, \frac{9x}{x + y + z} \right)$ and $g(x, y) = (y - x, 2xy^2)$. Find the derivative of the composition $g \circ f$ at the point $(1, 1, 1)$.

(13) Let $z = f(x, y)$ be a function of two variables and let $x = (u^2 - v^2)/2$ and $y = uv$ be change-of-variable equations. Calculate $z_u$ and $z_v$ in terms of $f_x$, $f_y$, $u$, and $v$.

(14) Let $f(x, y) = y + 3x^2$. Verify by direct calculation that the gradient vector $\nabla f(1, -1)$ is perpendicular to the level curve $f(x, y) = 2$ at $(1, -1)$. Sketch the level curve near $(1, -1)$ and the gradient vector at $(1, -1)$.

(15) (a) Find the stationary points of the function $f(x, y) = xy$ and classify them.

(b) Find the global maximum and minimum of $f(x, y)$ on the region $x^2 + y^2 \leq 8$.

(16) (a) Find the stationary points of the function $f(x, y) = 2x^2 + x + y^2 - 2$ and classify them.

(b) Find the global maximum and minimum of $f(x, y)$ in the region $x^2 + y^2 \leq 4$.

(17) Let $f(x, y) = xy^2 - 5x^2y$. Find the maximum and minimum value of $f$ on the region above the graph of $y = x^2$ and below the line $y = 4$.

(18) Your boss wants you to build a rectangular box. He gives you $9 to buy the materials to build the box, and he wants the bottom to be stronger. The material for the bottom costs $2/ft, and the material for the sides and the top costs $1/ft. What are the dimensions of the box with the largest volume that you can build with the $9 your boss gave you?

(19) A shipping box with two dividers is to be constructed from 48 m² of cardboard. What are the dimensions of the box with largest volume that can be constructed?

(20) Find the maximum volume of a rectangular box with no top that can be built with 1200 square feet of material.