(2) Let \( f : X \to Y \) be a function, and let \( A \subseteq Y \).

(a) Prove that \( f(f^{-1}(A)) \subseteq A \).

To prove that \( f(f^{-1}(A)) \subseteq A \), we need to prove that every element of \( f(f^{-1}(A)) \) is also an element of \( A \). Let \( y \in f(f^{-1}(A)) \). Then there exists \( x \in f^{-1}(A) \) such that \( f(x) = y \). Since \( x \in f^{-1}(A) \), we have that \( y = f(x) \in A \).

(b) True/False: If \( f \) is one-to-one, then \( f(f^{-1}(A)) = A \).

FALSE. Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = e^x \), and let \( A = [-1, \infty) \). Then \( f^{-1}(A) = \mathbb{R} \), and \( f(f^{-1}(A)) = f(\mathbb{R}) = (0, \infty) \neq A \).

(3) Let

\[
E = \left\{ \frac{6n + 1}{2n} \mid n \in \mathbb{N} \right\}.
\]

(a) Find \( \inf E \) and \( \sup E \).

\( \inf E = 3 \) and \( \sup E = 7/2 \).

(b) For \( \inf E \), prove that the number you found in part (a) is indeed the infimum of \( E \).

The sequence

\[
x_n = \frac{6n + 1}{2n} = 3 + \frac{1}{2n}
\]

converges to 3 as \( n \to \infty \). Moreover, it is a decreasing sequence because \( 1/(2n) \) is decreasing. By the Monotone Convergence Theorem, \( x_n \) converges to \( \inf E \), and therefore \( \inf E = 3 \).

(4) (a) Using the definition of convergence of a sequence, prove that, if \( x_n \to 4 \), then \( x_n^2 \to 16 \).

We need to prove that, for every \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( |x_n^2 - 16| < \varepsilon \) for all \( n \geq N \). Since we know that \( x_n \to 4 \), we want to relate \( |x_n^2 - 16| \) to \( |x_n - 4| \). The relation is

\[
|x_n^2 - 16| = |x_n - 4||x_n + 4|.
\]

The idea of the proof is that the first term on the right hand side of the equation is \( < \varepsilon \) if \( n \) is “big enough” and the second term is bounded because \( x_n \) converges. Here is a precise proof. Since \( x_n \to 4 \), \( \{x_n\} \) is bounded. Therefore, \( \{x_n + 4\} \) is bounded, and there exists a constant \( C > 0 \) such that \( |x_n + 4| < C \) for all \( n \in \mathbb{N} \). Since \( x_n \to 4 \), for \( \varepsilon/C > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_n - 4| < \varepsilon/C \) for all \( n \geq N \). Therefore,

\[
|x_n^2 - 16| = |x_n - 4||x_n + 4| < \frac{\varepsilon}{C} \cdot C = \varepsilon
\]

for all \( n \geq N \).

(b) True/False: If \( x_n \to 4 \), then \( \sqrt{x_n} \) is well-defined for \( n \) big enough (i.e., there exists \( N \in \mathbb{N} \) such that \( x_n \geq 0 \) for all \( n \geq N \)), and \( \sqrt{x_n} \to 2 \).

TRUE. Since \( x_n \to 4 \), \( x_n \) is a number close to 4 for \( n \) big enough, and therefore \( x_n \geq 0 \) if \( n \) is big enough. Moreover, since \( x_n \geq 0 \), \( \sqrt{x_n} \) is well-defined if \( n \) is big enough. As \( x_n \) approaches 4, \( \sqrt{x_n} \) approaches \( \sqrt{4} = 2 \).

(5) True/False: If a sequence \( \{x_n\} \) satisfies the condition that \( x_{n+1} = 1 + \sqrt{x_n - 1} \) for all \( n \in \mathbb{N} \), then \( x_n \to 2 \).

FALSE: Let \( x_1 = 1 \). Then \( x_2 = 1 + \sqrt{1-1} = 1 \), \( x_3 = 1 + \sqrt{1-1} = 1 \), and so on. For each \( n \), \( x_n = 1 + \sqrt{1-1} = 1 \), and \( x_n \to 1 \).