Note: I have actually decided to write the complete solution of each problem.

(1) Use the Principle of Mathematical Induction to prove that, for all \( n \in \mathbb{N} \),
\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2.
\]

Solution. \( P(1) \): 1 = 1 is true.
Assume \( P(n) \) and prove \( P(n+1) \): \( P(n) \) says that
\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2.
\]
Adding \( 2(n + 1) - 1 = 2n + 1 \) to both sides, we obtain that
\[
1 + 3 + 5 + \cdots + (2n - 1) + 2(n + 1) - 1 = n^2 + 2n + 1 = (n + 1)^2,
\]
which is \( P(n+1) \).

(2) True/False: If \( A, B \subseteq \mathbb{R} \) are bounded, then \( \sup(A \cap B) \leq \sup A, \sup B \).

Solution. True. For all \( x \in A \cap B \), we have that \( x \in A \), and therefore \( x \leq \sup A \),
because \( \sup A \) is an upper bound for \( A \). Similarly, \( x \leq \sup B \) for all \( x \in A \cap B \).
This means that \( \sup A \) and \( \sup B \) are upper bounds for \( A \cap B \). By definition of suprema of \( A \cap B \),
\( \sup(A \cap B) \leq \sup A, \sup B \).

(3) Let \( E \subseteq \mathbb{R} \) be bounded, and let \( k \) be a negative real number. Prove that
\[
\inf(kE) = k \sup E,
\]
where \( kE = \{kx \mid x \in E\} \).

Solution. By definition of supremum, we have that
(a) \( \sup E \) is an upper bound for \( E \).
(b) For any upper bound \( M \) of \( E \), \( \sup E \leq M \).
We need to prove that \( k \sup E \) is the infimum of the set \( kE \), i.e., we need to prove that
(c) \( k \sup E \) is a lower bound for \( kE \).
(d) For any lower bound \( N \) of \( kE \), \( k \sup E \geq N \).
We shall prove this by showing that (a)\(\Rightarrow\)(c) and (b)\(\Rightarrow\)(d).

Part I: (a)\(\Rightarrow\)(c). (a) says that \( x \leq \sup E \) for all \( x \in E \). Multiplying by \( k < 0 \)
on both sides, we obtain that \( kx \geq k \sup E \) for all \( x \in E \). Therefore, \( k \sup E \) is a lower bound for \( kE \).

Part II: (b)\(\Rightarrow\)(d). Let \( N \) be a lower bound for \( kE \). We want to show that
\( k \sup E \geq N \). Since \( N \) is a lower bound for \( kE \), \( kx \geq N \) for all \( x \in E \). Multiplying by \( 1/k < 0 \) on both sides, we obtain that \( x \leq N/k \) for all \( x \in E \), i.e., \( N/k \) is an upper bound for \( E \). Therefore, by (b), \( \sup E \leq N/k \). Multiplying by \( k < 0 \) on both sides, we obtain that \( k \sup E \geq N \).
(4) Let
\[ E = \left\{ \frac{n}{4n+1} \left| n \in \mathbb{N} \right. \right\} \].

(a) Find \( \inf E \) and \( \sup E \). [No proof required here.]

Solution. \( \inf E = 1/5, \sup E = 1/4. \)

(b) For \( \sup E \), prove that the number you found in part (a) is indeed the supremum of the set \( E \).

Solution. To show that \( 1/4 \) is the supremum of \( E \), we need to show two things:

(I) \( 1/4 \) is an upper bound for \( E \).

(II) For any upper bound \( M \) of \( E \), \( 1/4 \leq M \).

(I) For all \( n \in \mathbb{N} \),
\[ \frac{n}{4n+1} \leq \frac{1}{4}. \]
Indeed, this is equivalent to the statement that \( 4n \leq 4n+1 \), which is true for all \( n \in \mathbb{N} \).

(II) Let \( M \) be an upper bound for \( E \), and assume by contradiction that \( 1/4 > M \).

Let \( \varepsilon = 1/4 - M \).

Since
\[ \frac{n}{4n+1} = \frac{1}{4 + \frac{1}{n}} \to \frac{1}{4} \]
as \( n \to \infty \), there exists \( N \in \mathbb{N} \) such that
\[ \left| \frac{n}{4n+1} - \frac{1}{4} \right| < \varepsilon \]
for \( n \geq N \). By (I), we have that
\[ \left| \frac{n}{4n+1} - \frac{1}{4} \right| = \frac{1}{4} - \frac{n}{4n+1} \],
and this is \( < \varepsilon = 1/4 - M \) if and only if
\[ M < \frac{n}{4n+1}. \]
This gives a contradiction, since \( M \) is an upper bound for \( E \).

(5) Find the set of numbers \( x \in \mathbb{R} \) for which there exists \( \varepsilon > 0 \) such that \( x < -\varepsilon \) or \( x > \varepsilon \).

Solution. It is the set \((-\infty, 0) \cup (0, \infty)\).
(6) Find
\[ \bigcup_{n \in \mathbb{N}} \{ x \in \mathbb{R} \mid |x - n| < n \}. \]

Solution. It is the set \((0, \infty)\).

(7) Let \(f : X \to Y\) be a function, and let \(A, B \subseteq X\).
(a) Prove that \(f(A \cap B) \subseteq f(A) \cap f(B)\).

Solution. Since \(A \cap B \subseteq A\), we have that \(f(A \cap B) \subseteq f(A)\). Similarly, since \(A \cap B \subseteq B\), we have that \(f(A \cap B) \subseteq f(B)\). Therefore, \(f(A \cap B) \subseteq f(A) \cap f(B)\).

(b) Provide an example where \(f(A \cap B) \neq f(A) \cap f(B)\).

Solution. Let \(f : \mathbb{R} \to \mathbb{R}\) be the function defined by \(f(x) = x^2\). Let \(A\) be the interval \([0, 2]\) and \(B\) be the interval \([-2, 1]\). Then \(A \cap B = [0, 1]\) and \(f(A \cap B) = [0, 1]\), while \(f(A) = [0, 4]\), \(f(B) = [0, 4]\), and \(f(A) \cap f(B) = [0, 4]\).

(c) State a condition on \(f\) under which \(f(A \cap B) = f(A) \cap f(B)\). [No proof required here.]

Solution. It is true if \(f\) is one-to-one.

(d) Give an idea of why the condition you gave in part (c) is enough to imply that \(f(A \cap B) = f(A) \cap f(B)\).

Solution. As we saw in part (a), \(f(A \cap B) \subseteq f(A) \cap f(B)\). Therefore, for them not to be equal, there has to exist an element \(y\) in \(f(A) \cap f(B)\) which is not in \(f(A \cap B)\), i.e., an element \(y\) which comes both from \(A\) and from \(B\), but not from \(A \cap B\). This cannot happen if \(f\) is 1-1, because then \(y\) can only come from 1 element, and therefore if it comes from both \(A\) and \(B\), it must come from \(A \cap B\).

(8) Let \(f : X \to Y\) be a function, and let \(A \subseteq X\).
(a) Prove that \(A \subseteq f^{-1}(f(A))\).

Solution. Let \(x \in A\). Then \(f(x) \in f(A)\). Since \(x\) maps to \(f(x)\) which is in \(f(A)\), we have that \(x \in f^{-1}(f(A))\).

(b) Provide an example where \(A \neq f^{-1}(f(A))\).

Solution. Let \(f : \mathbb{R} \to \mathbb{R}\) be the function defined by \(f(x) = x^2\). Let \(A\) be the interval \([0, 2]\). Then \(f(A) = [0, 4]\) and \(f^{-1}(f(A)) = [-2, 2]\).

(c) State a condition on \(f\) under which \(A = f^{-1}(f(A))\). [No proof required here.]

Solution. It is true if \(f\) is one-to-one.

(d) Give an idea of why the condition you gave in part (c) is enough to imply that \(A = f^{-1}(f(A))\).
Solution. The only way this statement could be false is if there is an element $x \notin A$ which maps to $f(A)$. But $f(A)$ is the set of all elements which come from $A$. If $f$ is one-to-one, each element of $f(A)$ can only come from one element, and that element must be in $A$ for its image to be in $f(A)$.

(9) Let $f: X \to Y$ be a function, and let $A, B \subseteq Y$.

True/False: If $f^{-1}(A) \subseteq f^{-1}(B)$, then $A \subseteq B$.

[If true, give a proof. If false, provide a counterexample.]

Solution. False. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Let $A$ be the interval $[-1, 1]$, and let $B$ be the interval $[0, 1]$. Then $f^{-1}(A) = [-1, 1]$ and $f^{-1}(B) = [-1, 1]$. Therefore, $f^{-1}(A) \subseteq f^{-1}(B)$ but $A \nsubseteq B$.

(10) Let $f: X \to Y$ be an invertible function (i.e., such that there exists a function $g: Y \to X$ such that $f \circ g$ is the identity and $g \circ f$ is the identity). Prove that $f$ is one-to-one and onto.

Solution. Part I: $f$ is one-to-one. Let $x_1, x_2 \in X$. We need to prove that, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Starting with $f(x_1) = f(x_2)$ apply $g$ to both sides to obtain $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is the identity function, we have that $x_1 = x_2$.

Part II: $f$ is onto. To prove that $f$ is onto, we need to prove the following statement: For every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Let $y \in Y$, and take $x = g(y)$. Then $f(x) = f(g(y)) = y$ because $f \circ g$ is the identity function. Therefore, for each $y \in Y$, we found an element $x$ such that $f(x) = y$.

(11) Using (only) the definition of convergence of a sequence, prove that

$\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$.

Solution. Fix an $\varepsilon > 0$, and pick an integer $N > 1/\varepsilon$. Then

$\left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

for all $n \geq N$.

(12) Using (only) the definition of divergence of a sequence, prove that

$\lim_{n \to \infty} \frac{n^2}{n + 1} = \infty$. 
Solution. Fix $M > 0$, and pick an integer $N > M + 1$. We claim that

\[
\frac{n^2}{n + 1} > M.
\]

Indeed, we are going to have an equality if and only if $n^2 = M(n + 1)$. Solving this for $n$, we get two solutions

\[
n = \frac{M \pm \sqrt{M^2 + 4M}}{2}.
\]

We get our inequality (1) above as long as we have $n$ bigger than both of the solutions. Taking $n \geq N > M + 1$ suffices.

(13) (a) True/False: If $x_n \to a$, then $|x_n| \to |a|$.

Solution. True. Fix an $\varepsilon > 0$. Since $x_n \to a$, there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. Therefore, using the last Triangle Inequality from Theorem 1.7,

\[
|x_n| - |a| \leq |x_n - a| < \varepsilon
\]

for all $n \geq N$, and $|x_n| \to |a|$.

(b) True/False: If $|x_n| \to a$, then $x_n \to b$, where $a = |b|$.

Solution. False. Let $x_n = (-1)^n$. Then $|x_n| = 1 \to 1$, while $x_n$ does not converge.

(14) (a) True/False: If $x_n \to 0$ and $\{y_n\}$ is bounded, then $x_ny_n \to 0$.

Solution. True. Since $\{y_n\}$ is bounded, there exists $C > 0$ such that $|y_n| \leq C$. Fix an $\varepsilon > 0$. Since $x_n \to 0$, and $\varepsilon/C > 0$, there exists $N \in \mathbb{N}$ such that

\[
|x_n| < \frac{\varepsilon}{C}
\]

for all $n \geq N$. Then

\[
|x_ny_n| = |x_n||y_n| \leq C|x_n| < \varepsilon
\]

for all $n \geq N$, and $x_ny_n \to 0$.

(b) True/False: If $x_n \to \infty$ and $\{y_n\}$ is bounded, then $x_ny_n \to \infty$.

Solution. False. Let $x_n = n$ and $y_n = (-1)^n$. Then $x_ny_n$ does not converge.

(15) Suppose that $0 \leq x_1 \leq 2$, and define $x_{n+1} = \sqrt{2 + x_n}$.

(a) Using the Principle of Mathematical Induction, prove that $x_n$ is an increasing sequence bounded above.

Solution. Let $P(n)$ be the statement “$0 \leq x_n \leq 2$ and $x_{n+1} \geq x_n$.”

$P(1)$ is the statement “$0 \leq x_1 \leq 2$ and $x_2 \geq x_1$.” The first part of $P(1)$ is true by hypothesis, but we need to prove that $x_2 \geq x_1$, i.e., $\sqrt{2 + x_1} \geq x_1$. Since $x_1 > 0$, we have that $\sqrt{2 + x_1} \geq x_1$ is equivalent to $2 + x_1 \geq x_1^2$, i.e., $x_1^2 - x_1 - 2 \leq 0$, i.e.,.
\((x_1 - 2)(x_1 + 1) \leq 0\). Since \(0 \leq x_1 \leq 2\), \(x_1 - 2 \leq 0\) and \(x_1 + 1 > 0\). Therefore, \((x_1 - 2)(x_1 + 1) \leq 0\), and \(P(1)\) is true.

Assume now \(P(n)\) and prove \(P(n+1)\). We then know that \(0 \leq x_n \leq 2\) and \(x_{n+1} \geq x_n\), and we need to prove that \(0 \leq x_{n+1} \leq 2\) and \(x_{n+2} \geq x_{n+1}\). Now, \(x_{n+1} = \sqrt{2 + x_n}\). It is clearly \(\geq 0\) because \(x_n \geq 0\) implies that \(2 + x_n > 0\), and therefore \(\sqrt{2 + x_n} > 0\). Moreover, it is \(\leq 2\) because \(x_n \leq 2\) implies that \(2 + x_n \leq 4\), and therefore \(\sqrt{2 + x_n} \leq 2\). The proof that \(x_{n+2} \geq x_{n+1}\) is just like the proof of \(x_2 \geq x_1\) that we did above, replacing \(x_2\) with \(x_{n+2}\) and \(x_1\) with \(x_{n+1}\).

(b) Prove that \(\{x_n\}\) converges.

This follows from the Monotone Convergence Theorem.

(c) Find the limit of \(\{x_n\}\) as \(n \to \infty\).

Let \(L\) be the limit. Since \(x_n \to L\), we also have that \(x_{n+1} \to L\). Starting with \(x_{n+1} = \sqrt{2 + x_n}\), and taking the limit on both sides, we obtain that \(L = \sqrt{2 + L}\). Therefore, \(L^2 = 2 + L\), i.e., either \(L = -1\) or \(L = 2\). Since \(x_n \geq 0\), we cannot have that \(L = -1\), and therefore the limit is 2.