MATH 2210 - Test 3 Sample Questions

Note: For full credit, explain your work and justify your answers.

(1) Calculate the integral \( \int_C (x + y) \, ds \), where \( C \) is the line segment from (0, 2) to (-1, 1).

(2) Calculate the integral \( \int_C 2xy \, dx + (x^2 - y) \, dy \), where \( C \) is the line segment from (1, 1) to (0, 3).

(3) Calculate the work done in the vector field \( \vec{F} = \langle 2xy^2, 2x^2y + z, y + x^2 \rangle \) to move an object around the square with vertices (0, 1, 0), (1, 1, 0), (1, 1, 1), and (0, 1, 1).

(4) Calculate the flow of the vector field \( \vec{F} = \langle x + y, x + y \rangle \) through the curve \( C \) given by the portion of the graph of \( y = x^2 \) between (0, 0) and (2, 4).

(5) Calculate the integral
\[
\int_C 5x \, dx + 6x^2z \, dx + y^2 \, dy + 2yz \, dy + y^2 \, dz + \sin(x^2y) \, dz + 2x^3 \, dz,
\]
where \( C \) is the circle of center \((0, 0, 2)\) and radius 15 in the plane \( z = 2 \).

(6) Prove that \( \text{div} (\text{curl} \vec{F}) = 0 \) for every vector field \( \vec{F} \).

(7) Prove that \( \text{curl} (\text{grad} (f)) = \vec{0} \) for every function \( f \).

(8) Calculate the integral \( \int_C 12x \, dx + xy^2 \, dy \), where \( C \) is the circle with center \((0, 0)\) and radius 2.

(9) Let \( C \) be the circle of radius 2 and center \((0, 1, 0)\) in the \( y, z \)-plane. Show that
\[
\int_C 4xy \, dx + e^{xy^2} \, dx - y^2 \, dx - 2x^2y \, dy + y \, dz - x \, dz = \int_C 2xy^2 \, dx + z \, dx + 2xy \, dy - 2x^2 \, dy - z \, dy.
\]

(10) Let \( f(x, y, z) \) be a function. Calculate \( \text{div} (\text{grad} (f)) \) in terms of \( f \) and its partial derivatives.
Formula Sheet

- **Line Integral**
  \[ \int_C f \, ds = \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt. \]

- **Work**
  \[ W = \int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_C P \, dx + Q \, dy, \]
  where \( \vec{F} = \langle P, Q \rangle \) is a vector field.

- **Work in 3-Space**
  \[ W = \int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_C P \, dx + Q \, dy + R \, dz, \]
  where \( \vec{F} = \langle P, Q, R \rangle \) is a vector field.

- **Flow**
  \[ \int_C \vec{F} \cdot \vec{N} \, ds = \int_C -Q \, dx + P \, dy, \]
  where \( \vec{F} = \langle P, Q \rangle \) is a vector field.

- **Fundamental Theorem of Line Integrals** If a vector field \( \langle L, M \rangle \) is the gradient of a function \( f \) (which means, \( L = f_x \) and \( M = f_y \)), then
  \[ \int_C L \, dx + M \, dy = f(B) - f(A), \]
  where \( A \) is the initial point of \( C \) and \( B \) is the final point of \( C \).

- **Fundamental Theorem of Line Integrals in 3-Space** If a vector field \( \langle L, M, N \rangle \) is the gradient of a function \( f \) (which means, \( L = f_x \), \( M = f_y \), and \( N = f_z \)), then
  \[ \int_C L \, dx + M \, dy + N \, dz = f(B) - f(A), \]
  where \( A \) is the initial point of \( C \) and \( B \) is the final point of \( C \).

- **Green’s Theorem** If \( C \) is a closed curve (which means, the intial point \( A \) is the same as the final point \( B \) of the curve \( C \)), then
  \[ \int_C L \, dx + M \, dy = \iint_R (M_x - L_y) \, dA, \]
  where \( R \) is the region inside the closed curve \( C \).

- **Divergence** If \( \vec{F} = \langle P, Q, R \rangle \) is a vector field, then
  \[ \text{div}(\vec{F}) = \nabla \cdot \vec{F} = P_x + Q_y + R_z. \]

- **Curl** If \( \vec{F} = \langle P, Q, R \rangle \) is a vector field, then
  \[ \text{curl}(\vec{F}) = \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle. \]