MATH 2210 - All Test Questions

1. Consider the vectors \( \vec{a} = (-3, 2, 1) \) and \( \vec{b} = (2, -1, 5) \).
   (a) Find \( \vec{a} \cdot \vec{b} \).
   (b) Find \( \vec{a} \times \vec{b} \).
   (c) Find the projection of \( \vec{a} \) in the direction of \( \vec{b} \).

2. (a) Find the matrix of the linear function \( f: \mathbb{R}^2 \to \mathbb{R} \) defined by \( f(x, y) = 3x - 2y \).
   (b) Find the matrix of the linear function \( g: \mathbb{R} \to \mathbb{R}^2 \) defined by \( g(x) = (-2x, 5x) \).
   (c) Find the matrix of the composition \( f \circ g \).
   (d) Find the matrix of the composition \( g \circ f \).

3. Sam is trying to throw a ball and hit a 5 m target which is 20 m away (horizontally). He throws the ball with a speed of 20 m/s and at an angle of 60° from the horizontal, does he hit the target? (Remember that acceleration caused by gravity is 9.8 m/s², and that \( \sin(60°) = \sqrt{3}/2 \), \( \cos(60°) = 1/2 \))

4. Find the equation of the plane that contains both the line with equations
   \[ \frac{x - 3}{2} = \frac{y + 2}{4} = \frac{z - 1}{4} \]
   and the line parametrized by
   \[ \vec{r}(t) = (-3, 1, -5) + t(2, -1, 2). \]

5. Let \( A \) be a 3 by 3 matrix, and let \( \vec{a}, \vec{b}, \vec{c} \) be the three rows of \( A \) (each is a vector with 3 entries). Show that, if \( \vec{b} \) is a multiple of \( \vec{a} \), then the determinant of \( A \) is zero.

6. (Extra-credit) Let \( A \) be the 2 by 2 matrix
   \[ A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}. \]
   Find all the numbers \( a \) and all non-zero vectors \( \vec{v} \) such that
   \[ A \vec{v} = a \vec{v}. \]

7. Calculate the following limits.
   (a) \( \lim_{(x, y) \to (0, 0)} \frac{2xy^2}{x^2 + y^2} \).
   (b) \( \lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^3 + y^4} \).

8. Let \( f(x, y) = 2x^2 y^3 - 3x e^y + x \ln(x) - \sin(xy) \).
   (a) Find the partial derivatives \( f_x \) and \( f_y \).
   (b) Find the gradient of \( f \) at the point \((1, 0)\).
   (c) Show by direct calculation that the gradient is perpendicular to the level curve that contains the point \((1, 0)\).

9. Consider the graph of the function \( f(x, y) = 2x^2 \ln(xy) + 3\sqrt{y} \). Find the tangent plane at the point \((1, 1)\).

10. Consider the functions
    \[ f: \mathbb{R}^3 \to \mathbb{R}^2 \quad f(x_1, x_2, x_3) = \left( x_1^2 x_3, 3x_1 + \frac{x_2}{x_3} \right) \]
    \[ g: \mathbb{R}^2 \to \mathbb{R}^2 \quad g(y_1, y_2) = (\sqrt{y_1}, 3y_1^2 - 2y_1 e^{y_1} y_2) \].
    Find the derivative of \( g \circ f \) at the point \((1, -3, 1)\).
11. Show that, if \( f: \mathbb{R}^2 \to \mathbb{R} \) is a function and \( P \) is a point, then the gradient \( \nabla f(P) \) of \( f \) at \( P \) is equal to \( \langle Df_{(0,0)}(P), Df_{(1,0)}(P) \rangle \). (Remember that \( Df_u(P) \) denotes the directional derivative of \( f \) at the point \( P \) in the direction \( u \)).

12. (Extra-credit) Let \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) be a linear function. Show that the derivative of \( f \) at a point \( P \) is the linear function \( f \) itself. In other words, show that \( \text{df}_P = f \).

13. Consider the function \( f(x, y) = xy \).
   
   (a) Find the stationary points and classify them.
   
   (b) Find the global maximum and minimum on the region \( x^2 + y^2 \leq 8 \).

14. Calculate the double integral \( \iint_D 2xy \, dA \) where \( D \) is the triangle with vertices \((0, 0), (1, 1), \) and \((2, 0)\).

15. Set up the triple integral \( \iiint_K f(x, y, z) \, dV \), where \( K \) is the region delimited by the planes \( y = 0, y = x, z = 0 \) and \( x + y + 2z = 2 \).

16. Your boss wants you to build a rectangular box. He gives you \$9 to buy the materials to build the box, and he wants the bottom to be stronger. The material for the bottom costs \$2/\text{ft}^2\), and the material for the sides and the top costs \$1/\text{ft}^2. What are the dimensions of the box with the largest volume that you can build with the \$9 your boss gave you?

17. Consider two functions \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) of one variable.
   
   \[
   \int_0^1 \int_0^x (f''(x) - g''(y)) \, dy \, dx = f(0) + g(0) + f'(1) + g'(0) - f(1) - g(1).
   
   (Extra-credit) Use spherical coordinates to calculate
   
   \[
   \iiint_K \sqrt{x^2 + y^2} \, dV,
   
   \text{where } K \text{ is the sphere of radius 1 centered at } (0, 0, 0).
   
19. Calculate the following integrals.
   
   (a) \( \int_C (x + y) \, ds \), where \( C \) is the line segment from \((0, 2)\) to \((-1, 1)\).
   
   (b) \( \int_C 2xy \, dx + (x^2 - y) \, dy \), where \( C \) is the line segment from \((1, 1)\) to \((0, 3)\).

20. Calculate the work done in the vector field \( \vec{F} = <2xy^2, 2x^2y + z, y + x^2> \) to move an object around the square with vertices \((0, 1, 0), (1, 1, 0), (1, 1, 1), \) and \((0, 1, 1)\).

21. Calculate the flow of the vector field \( \vec{F} = <x + y, x + y> \) through the curve \( C \) given by the portion of the graph of \( y = x^2 \) between \((0, 0)\) and \((2, 4)\).

22. Calculate the integral \( \int_C 5x \, dx + 6x^2z \, dx + y^2 \, dy + 2yz \, dy + y^2 \, dz + \sin(x^2y) \, dz + 2x^3 \, dz \), where \( C \) is the circle of center \((0, 0, 2)\) and radius 15 in the plane \( z = 2 \).

23. (a) Prove that \( \text{div}(\text{curl}(\vec{F})) = 0 \) for every vector field \( \vec{F} \).

   (b) Prove that \( \text{curl}(\text{grad}(f)) = \vec{0} \) for every function \( f \).

24. (Extra-credit) Calculate the integral \( \int_C 12x \, dx + xy^2 \, dy \), where \( C \) is the circle with center \((0, 0)\) and radius 2.