Section 5.2, Normal Distributions: Finding Probabilities

Section 5.3, Normal Distributions: Finding Values

In the last section, we focused on finding probabilities for the standard normal distribution. Now, we will consider normal distributions with $\mu \neq 0$ and/or $\sigma \neq 1$.

To find probabilities for variables from a normal distribution with mean $\mu$ and standard deviation $\sigma$, we will first need to find the $z$-score to convert to standard normal, then we can use the table that we used in the last section to find probabilities.

We will also look at going from a probability to a range of observations that correspond to that probability. For example, we will look at finding quartiles and percentile. The $p^{th}$ percentile is a value such that $p$ percent of the observations fall at or below that value.

Examples

1. Find the $7^{th}$ percentile for a standard normal distribution. We want to find $x$ such that $P(z \leq x) = 0.07$. The decimal on the table closest to .07 is .0694 (corresponding to a $z$-score of $-1.48$), so the $7^{th}$ percentile is approximately $-1.48$.

2. ACT scores are approximately normal with a mean of 21 and a standard deviation of 4.7.

   (a) What is the probability that a student chosen at random receives less than a 26 on the ACT? Let $x$ represent this student’s ACT score. The $z$-score for 26 is $z = \frac{26 - 21}{4.7} = 1.06$ (Note: You should round to 2 decimal places, since the table goes to 2 decimal places.). So, $P(x < 26) = P(z < 1.06) = 0.8554$.

   (b) What is the probability that a student chosen at random receives at least a 24 on the ACT? The $z$-score for 24 is $z = \frac{24 - 21}{4.7} = 0.64$, so $P(x \geq 24) = P(z \geq 0.64) = 1 - P(z < 0.64) = 1 - 0.7389 = 0.2611$.

   (c) What is the third quartile for the ACT? We want to find $y$ such that $P(x < y) = 0.75$. The closest probability to 0.75 on the table is 0.7486, which corresponds to a $z$-score of 0.67. Now, we want the $y$-value that corresponds to a $z$-score of 0.67. We can do this by solving the equation:

   $$\frac{y - 21}{4.7} = 0.67$$
   $$y - 21 = 3.149$$
   $$y = 24.149,$$

   so the third quartile is approximately 24.149.

   (d) What is the 80$^{th}$ percentile for the ACT? We want to find $y$ such that $P(x < y) = 0.8$. The closest probability to 0.8 on the table is 0.7995, which corresponds to a $z$-score of 0.84. Again, we want the $y$-value that corresponds to a $z$-score of 0.84. We can do this by solving the equation:

   $$\frac{y - 21}{4.7} = 0.84$$
   $$y - 21 = 3.948$$
   $$y = 24.948,$$

   so the third quartile is approximately 24.948.
3. A brand of cassette decks had a deck life that was normally distributed with a mean of 2.3 years and a standard deviation of 0.4 years.

   (a) What is the probability that the cassette deck will break down less than one year after purchase?

\[ P(x < 1) = P \left( z < \frac{1 - 2.3}{0.4} \right) = P(z < -3.25) = 0.0006 \]

   (b) What is the probability that the cassette deck will last for at least 3 years after purchase?

\[ P(x \geq 3) = 1 - P(x < 3) = 1 - P \left( z < \frac{3 - 2.3}{0.4} \right) = 1 - P(z < 1.75) = 1 - 0.9599 = 0.0401 \]

   (c) What is the 99\textsuperscript{th} percentile for the life of these cassette decks?

We want \( y \) such that \( P(x \leq y) = 0.99 \). The closest probability on the table is 0.9901, occurring at \( z = 2.33 \), so

\[
\begin{align*}
y - 2.3 &= 2.33 \\
y - 2.3 &= 0.932 \\
y &= 3.232,
\end{align*}
\]

so the 99\textsuperscript{th} percentile is approximately 3.232 years.

4. A pizza parlor franchise specifies that the average amount of cheese on a large pizza should be 8 ounces and the standard deviation should be 0.5 ounces.

   (a) What is the probability that a pizza chosen at random has less than 7.3 ounces of cheese?

\[ P(x < 7.3) = P \left( z < \frac{7.3 - 8}{0.5} \right) = P(z < -1.4) = 0.0808 \]

   (b) What is the probability that a pizza has more than 8.95 ounces of cheese?

\[ P(x > 8.95) = 1 - P(x \leq 8.95) = 1 - P \left( z \leq \frac{8.95 - 8}{0.5} \right) = 1 - P(z \leq 1.9) = 1 - 0.9713 = 0.0287 \]

   (c) What is the probability that a pizza contains between 7.6 and 8.3 ounces of cheese?

\[ P(7.6 \leq x \leq 8.3) = P \left( \frac{7.6 - 8}{0.5} \leq z \leq \frac{8.3 - 8}{0.5} \right) = P(-0.8 \leq z \leq 0.6) = 0.7257 - 0.2119 = 0.5138 \]

   (d) What is the probability that a pizza has exactly 8 ounces of cheese?

Since this is a continuous distribution, the probability of taking an exact value is zero. (We can see this by noticing that \( P(x = 8) = P(8 \leq x \leq 8) \), so when we convert to \( z \)-scores and look up probabilities in the table, we'll be subtracting the same number from itself.)

   (e) What is the least amount of cheese that can be on a pizza that will still place in the top 10\% of cheesiest pizzas?

We want \( y \) such that \( P(x > y) = 0.1 \), which is the same a \( y \) that satisfies \( P(x < y) = 0.9 \).
0.8997 is the closest probability to 0.9 on the table, so $z = 1.28$. Then, finding $y$, we solve:

\[
\begin{align*}
\frac{y - 8}{0.5} &= 1.28 \\
y - 8 &= 0.64 \\
y &= 8.64
\end{align*}
\]

(f) Between what two values does the middle 70% of cheese lie?
We first want $y$ such that $P(-y < z < y) = 0.7$, and we will convert to the $x$ value later. By symmetry, we will have 15% of the observations above $y$, and 15% below $-y$. This means that 85% of the observations are below $y$, so $y$ satisfies $P(x < y) = 0.85$. 0.8508 is the closest probability on the table to this, which corresponds to a $z$-score of 1.04. Changing to a normal distribution with mean 8 and standard deviation 0.5, we get that the middle 70% of cheese lies between $8 - 1.04 \cdot 0.5 = 7.48$ and $8 + 1.04 \cdot 0.5 = 8.52$. 