Section 9.3, Average and Instantaneous Rates of Change: The Derivative

1 Average Rate of Change

The average rate of change of a function \( y = f(x) \) from \( x = a \) to \( x = b \) is:

\[
\frac{f(b) - f(a)}{b - a}.
\]

Note that this equals the slope of the line connecting the points \((a, f(a))\) and \((b, f(b))\).

Example Find the average rate of change of \( f(x) = x^2 - 2x + 4 \) on the interval \([1, 3]\).

\[
\frac{f(3) - f(1)}{3 - 1} = \frac{7 - 3}{2} = 2
\]

2 Derivative

If \( y = f(x) \) is a function, then the derivative of \( f(x) \) at any value of \( x \), denoted \( f'(x) \), is

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},
\]

if this limit exists. If \( f'(c) \) exists, we say that \( f \) is differentiable at \( c \).

If \( y = f(x) \), alternative notation for \( f'(x) \) includes \( y', \frac{dy}{dx}, \frac{d}{dx} f(x), D_x y, \) and \( D_x f(x) \).

The derivative has a variety of interpretations. First, \( f'(c) \) is the instantaneous rate of change of the function \( f \) at \( x = c \). Secondly, \( f'(c) \) is the slope of the line tangent to the graph of \( y = f(x) \) at \( x = c \). We will explore other uses of the derivative later this semester.

Examples

1. Find the instantaneous rate of change of \( f(x) = 2x - 4 \) at \( x = -1 \).

   We need to find \( f'(-1) \). From the definition of the derivative, with \( x = -1 \),

   \[
f'(-1) = \lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h} = \lim_{h \to 0} \frac{2(-1 + h) - 4 - (2(-1) - 4)}{h} = \lim_{h \to 0} \frac{2h}{h} = 2
\]

2. Find the derivative of \( f(x) = x^2 - x + 3 \).

   \[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - (x + h) + 3 - (x^2 - x + 3)}{h} = \lim_{h \to 0} \frac{2xh + h^2 - h}{h} = \lim_{h \to 0} (2x + h - 1) = 2x - 1
\]

3. Let \( f(x) = x^2 - x + 3 \) again. Find the slope of the line tangent to the graph of \( y = f(x) \) at \( x = -3 \).

   This question is asking us to find \( f'(-3) \). Since we already calculated the derivative, we know that \( f'(x) = 2x - 1 \), so:

   \[
f'(-3) = 2(-3) - 1 = -7
\]
We will learn “shortcuts” to calculate derivatives later, but for this section, make sure that you use the definition of the derivative for your calculations.

3 Application: Velocity

The velocity of an object is the rate at which its position is changing with respect to time.

Example

The height of a ball thrown upward at a speed of 30 ft/s from a height of 15 feet after $t$ seconds is given by:

$$S(t) = 15 + 30t - 16t^2$$

Find the average velocity of the ball in the first 2 seconds after it is thrown.

We are asked for the average rate of change of the position of the ball between times 0 and 2. Using our formula for average rate of change,

$$\frac{S(2) - S(0)}{2 - 0} = \frac{11 - 15}{2} = -2 \text{ ft/s}$$