Some loose ends

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1. Lebesgue integration on $\mathbb{R}^n$
If \( f(x_1, \ldots, x_n) \) is a smooth function on \( \mathbb{R}^n \) with support in the open set \( X \), its integral is

\[
\int_X f(x) \, dx_1 \ldots dx_n,
\]

which can be explicitly calculated (rarely) by reducing it to one-dimensional integrals, where one can apply the fundamental theorem of calculus. If we make a change of variables \( x = h(y) \) where \( h \) is an invertible smooth function the integral becomes

\[
\int_{h^{-1}(X)} f(h(y)) \left| \frac{\partial x}{\partial y} \right| \, dy
\]

since \( x \in X \) if and only if \( y = h^{-1}(x) \) lies in \( h^{-1}(X) \). What is important here is that this formula involves the absolute value of the Jacobian determinant.
The change of variables formula in 1D might seem a bit paradoxical but it agrees with the usual rules of calculus. For example

\[ \int_{-\infty}^{\infty} f(x) \, dx = -\int_{-\infty}^{\infty} f(-y) \, dy = \int_{-\infty}^{\infty} f(-y) \, dy \]

The point is that this integral represents an integral of a measure.
2. Integration of forms on $\mathbb{R}^n$
If $\omega$ is an $n$-form on $\mathbb{R}^n$ it can be written as $f(x) \, dx_1 \wedge \ldots \wedge dx_n$ and then its integral is

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x) \, dx_1 \ldots \, dx_n$$

The point is that we have to first arrange the formula for $\omega$ so as to match the standard orientation of $\mathbb{R}^n$. 
3. Integration on oriented manifolds
Suppose $M$ to be an **oriented manifold**. We can cover it by coordinate patches $U_i$ embedded in $\mathbb{R}^n$ in such a way that the orientations all match that of $M$, and we can find a **partition of unity** $\varphi_i$ subordinate to this covering. Then $\varphi_i \omega$ may be identified with a compactly supported form $\omega_i$ on $\mathbb{R}^n$ and

$$\int_M \omega = \sum_i \int_{U_i} \omega_i.$$
4. **Integration on arbitrary manifolds**
Suppose now that $M$ is an arbitrary manifold of dimension $n$. At each point $m$ of $M$ we have the one-dimensional real vector space $\bigwedge^n T_x$. The fibre bundle $\tilde{M}$ of orientations on $M$ is the quotient of $\bigwedge^n T_x - \{0\}$ by the positive real numbers, a set of two elements. The space $\tilde{M}$ is a two-fold covering of $M$. The manifold $\tilde{M}$ is **orientable** if and only if this bundle has a section, which is to say that at each point we have a continuous choice of orientation. If it is orientable then we can integrate forms over $\tilde{M}$, but only after making a choice of orientation. Reversing the orientation will change the sign of the integral. So there is no canonical way to integrate forms on $\tilde{M}$.

There is, however, a canonical way to integrate something else, called a **density** or **twisted $n$-form**.
The covering $\tilde{M}$ has a conical involution, interchanging orientations at any point of $M$. The $n$-forms on $M$ may be identified with forms on $\tilde{M}$ that are invariant under this involution. Since changing orientation changes the sign of an integral of a form, the integral of such a form on $\tilde{M}$ is 0. A twisted $n$ form on $M$ is defined to be an $n$-form on $\tilde{M}$ that is taken to its negative by the involution. If $\tilde{\omega}$ is such a form on $\tilde{M}$ then by definition

$$\int_M \tilde{\omega} = \frac{1}{2} \int_{\tilde{M}} \tilde{\omega}$$

In other words, what is invariantly defined on an arbitrary manifold is the integral of a twisted $n$-form.
The twisted $n$-forms on a manifold are sections of a one-dimensional fibre bundle on $M$. The fibre at $x$ is the space of all maps $f$ from $\bigwedge^n T_x$ to $\mathbb{R}$ such that

$$f(cv) = |c|f(v)$$

On any manifold there always exists at least one twisted $n$-form that never vanishes.
5. Homogeneous fibre bundles
Suppose now that $G$ is a Lie group and $H$ a closed subgroup. If $(\sigma, U)$ is a finite-dimensional representation of $H$, then there is associated to it a fibre bundle over $H \backslash G$ whose fibre at any point is non-canonically equal to $U$. Geometrically it is the quotient of $U \times G$ by the group $H$ taking $(u, g)$ to $(\sigma(h)u, hg)$. The sections of this bundle over $H \backslash G$ are the functions

$$ f: G \longrightarrow U $$

such that $f(hg) = \sigma(h)f(g)$ for all $h$ in $H$ and $g$ in $G$. One representation of $H$ is that on the tangent space at 1 of $H \backslash G$, which may be identified with $\mathfrak{h} \backslash \mathfrak{g}$. The bundle to conjugation $\text{Ad}$ is the tangent bundle. Another is the one dimensional representation of $H$ taking

$$ h \longrightarrow |\det \text{Ad}_{\mathfrak{h} \backslash \mathfrak{g}}(h)|^{-1} $$

and the associated bundle is twisted $n$-forms.
Take $G = \text{SL}_2(\mathbb{R})$ and $H = P$. Here $p \backslash g = \bar{n}$ and the twisted $n$-forms correspond to the character

$$\delta_P: \begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \mapsto a^2$$

Since $a^2 > 0$ these do not differ from ordinary $n$-forms. This remains true for the spaces $\mathbb{P}^1(\mathbb{R}^n)$ with $n$ odd, but fails for the non-orientable cases with $n$ even.

At any rate, a smooth real twisted $n$-form on $P \backslash G$ may be identified a smooth function $f$ from $G$ to $\mathbb{R}$ such that $f(pg) = \delta_P(p)f(g)$. I write integration of twisted $n$-forms as

$$\int_{P \backslash G} \omega$$

Since $G = PK$, the quotient $P \backslash G$ may be identified with $K \cap P \backslash K$, and if we assign $K$ a total measure 1 integration on $P \backslash G$ may be identified with integration over $K$. 

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There is another way to put this. If $f$ is a smooth function of compact support on $G$, then

$$\overline{f}(g) = \int_P f(pg) \, d\ell_p$$

is a density on $P \setminus G$—$\overline{f}(pg) = \delta_P(p) \overline{f}(g)$. Then with suitable normalizations

$$\int_G f(g) \, dg = \int_{P \setminus G} \overline{f}(x)$$

$$= \int_K dk \int_P f(pk) \, d\ell_p$$

$$= \int_K dk \int_A \delta_P(a)^{-1} \, da \int_N f(nak) \, dn$$

since the integral with respect to $d\ell P$ can also be expressed as

$$\int_A \delta_P(a)^{-1} \, da \int_N f(na) \, dn.$$
There is another formula for integration over $P \backslash G$. The set $P \overline{N}$ is open in $G$, and the integral

$$\int_{\overline{N}} f(\overline{n}) \, d\overline{n}$$

converges. It is, up to a constant, another valid formula. If we identify $\overline{N}$ with $\mathbb{R}$, what is the constant?
6. The smooth principal series
Any character (continuous homomorphism into $\mathbb{C}^\times$) of $A$ is of the form

$$
\chi_{s,m} : \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \mapsto |x|^s \text{sgn}^m(x)
$$

for some $s$ in $\mathbb{C}$ and $m = 0, 1$. This will be a unitary character if and only if $s = it$ for some real number $t$.

Any character of $A$ determines one of $P$ since $P/N = A$. Any continuous irreducible representation of $P$ is of this form (in particular trivial on $N$). In any continuous finite-dimensional representation of $P$ the subgroup $N$ is taken to unipotent matrices.

The **principal series** representations of $G$ are those induced from characters of $P$. 
\[ \text{Ind}^\infty (\chi | P, G) = \{ f \in C^\infty (G, \mathbb{C}) \mid f(pg) = \delta_P^{1/2} \chi(p)f(g) \text{ for all } p \in P, g \in G \} \]

The group \( G \) acts by the right regular action:

\[ R_g f(x) = f(xg) \]

- \( \text{Ind}^\infty (\delta_P^{-1/2}) = C^\infty (P \backslash G) \)
- \( \text{Ind}^\infty (\delta_P^{+1/2}) = \Omega^\infty (P \backslash G) \)
- \( \text{Ind}^\infty (\chi^{-1}) = \text{the dual of } \text{Ind}^\infty (\chi) \)

\[ \langle f, \varphi \rangle = \int_{P \backslash G} f(x)\varphi(x) \, dx \]

- \( \text{Ind}^\infty (\chi) \) is unitary if \( \chi \) is.
The best way to picture $\text{Ind}^\infty(\chi)$ is to describe its restriction to $K$.

Restricting $f$ to $K$ determines a map from $K$ to $\mathbb{C}$ such that

$$f(pk) = \chi(p)f(k)$$

for all $p$ in $P \cap K$. Because $G = PK$ this is an isomorphism. Since $P \cap K = \pm I$ and $\chi(-I) = (-1)^m$:

$$\text{Ind}^\infty(\chi) | K = \widehat{\sum}_{n \equiv m \mod 2} \varepsilon^n$$

where $\widehat{\sum}$ means a topological sum ($C^\infty$ Fourier series).
Let

\[ \varphi_n(pk) = \delta^{1/2} \chi(p) \varepsilon^n(k) \]

If

\[ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

then \( g = pk \) where

\[ p = \begin{bmatrix} 1/r & (ac + bd)/r \\ 0 & r \end{bmatrix} \quad (r = \sqrt{c^2 + d^2}) \]

\[ k = \begin{bmatrix} \gamma & -\sigma \\ \sigma & \gamma \end{bmatrix} \quad (\gamma = d/r, \sigma = c/r) \]

Therefore

\[ \varphi_n(g) = \delta^{1/2} \chi(1/r)(\gamma + i \sigma)^n \]
7. Explicit formulas
The Lie algebra $g$ acts on the subspace of finite sums of the $\varphi_n$. Recall the basis of the complex Lie algebra

$$
\kappa = \begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}
$$

$$
x_+ = \begin{bmatrix}
1 & -i \\
-\text{i} & 0 \\
\end{bmatrix}
$$

$$
x_- = \begin{bmatrix}
1 & \text{i} \\
\text{i} & 0 \\
\end{bmatrix}
$$

$$
[\kappa, x_\pm] = \pm 2i x_\pm
$$

so that

$$
\kappa \varphi_n = n \text{i} \varphi_n
$$

$$
\kappa(x_\pm \varphi_n) = x_\pm (\kappa \varphi_n) \pm 2i x_\pm \varphi_n
$$

$$
= (n \pm 2) \text{i} (x_\pm \varphi_n)
$$

$$
x_\pm \varphi_n = \textit{constant} \cdot \varphi_{n\pm 2}
$$
\[
x_{\pm} \varphi_n = \text{constant} \cdot \varphi_{n \pm 2}
\]

What is the constant? Since \( \varepsilon_n(1) = 1 \)

\[
x_{\pm} \varphi_n(1) = \text{constant}
\]

Here the Lie algebra acts on the right. So we use the basic trick (seen before).

\[
R_X f(g) = L_g X g^{-1} f(g)
\]

here with \( g = 1 \). Since

\[
x_{\pm} = \alpha + i(\kappa + 2\nu_+)
\]

\[
R_{x_{\pm}} \varepsilon_n(1) = [L_{\alpha \mp 2i\nu_+} \mp R_{i\kappa}] \varepsilon_n(1)
= (s + 1 \pm n)
\]
Summary:

\[
κ \varepsilon_n = nι \varepsilon_n
\]
\[
x_± \varepsilon_n = (s + 1 ± n)\varepsilon_n
\]

We have seen this before when \( s = -1 \) and \( s + 1 = 0 \) (except for some small change of sign) caused by a difference between left and right actions. The space of Harmonic functions is isomorphic to \( \text{Ind}(δ^{-1/2}) \). More generally:

Every irreducible \((g, K)\)-representation can be embedded into a principal series representation.

To be proven in a later lecture.
8. Intertwining operators
Some principal series are isomorphic to other principal series. Some principal series are reducible. To figure out what’s going on, we need to calculate the $G$-covariant (or $(g, K)$-covariant) maps from one principal series to another.

The start is a version of Frobenius reciprocity. I recall what this says for a finite group. Let $H$ be a subgroup of another group $G$. If $\sigma$ is an irreducible representation of $H$, we want to know how often an irreducible representation $\pi$ of $G$ occurs in the representation $I(\sigma)$ induced by $\sigma$. The answer is that $\pi$ occurs as often in $I(\sigma)$ as $\sigma$ occurs in the restriction of $\pi$ to $H$:

$$\dim \text{Hom}_G(\pi, I(\sigma)) = \dim \text{Hom}_H(\sigma, \pi)$$

But since representations of finite groups always decompose completely, this is also

$$\dim \text{Hom}_H(\pi, \sigma)$$
Theorem. (Frobenius reciprocity for finite groups) Suppose $H \subseteq G$ are finite groups. If $(\sigma, U)$ is any finite dimensional representation of $H$ and $(\pi, V)$ is one of $G$ then there is a canonical isomorphism

$$\text{Hom}_G(\pi, I(\sigma)) \cong \text{Hom}_H(\pi, \sigma)$$

$I(\sigma) = \{ f: G \to U | f(hg) = \sigma(h) \}$

Either side determines the other—$F_G(v) = F_H(\pi(g)v)$. 
Let

\[ \Lambda_1: \text{Ind}^\infty(\chi) \longrightarrow \mathbb{C}, \quad f \longmapsto f(1) \]

**Theorem.** *(Frobenius reciprocity for principal series)* If \( V \) is a smooth representation of \( G \) then composition with \( \Lambda_1 \) induces an isomorphism

\[ \text{Hom}(V, \text{Ind}^\infty(\chi|P,G)) = \text{Hom}_P(V, \delta^{1/2}\chi) \]

The Lie algebra \( \mathfrak{n} \) acts trivially on \( \mathbb{C} \), so any \( P \)-map from \( V \) to \( \delta^{1/2}\chi \) takes \( \nu_+\nu \) to 0. It must annihilate the subspace \( \mathfrak{n}V \) of \( V \) spanned all the \( \nu_+\nu \). In other words it must factor through the quotient \( V/\mathfrak{n}V \), on which \( A \) acts. So a new version of the theorem is

\[ \text{Hom}(V, \text{Ind}^\infty(\chi|P,G)) = \text{Hom}_A(V/\mathfrak{n}V, \delta^{1/2}\chi) \]

\[ = \text{Hom}_A(\chi^{-1}\delta^{-1/2}, \hat{V}[\mathfrak{n}]) \]
There are two kinds of $N$-invariant functionals on $\text{Ind}^\infty(\chi)$, corresponding to the two components in the Bruhat decomposition

$$G = P \cup PwN$$

Formally, we have the integral

$$\Lambda_w(f) = \int_N f(wn) \, dn$$

which satisfies

$$\Lambda_w(R_{n*} f) = \int_N f(wnn_*) \, dn = \Lambda_w(f)$$

...
... and then

\[ \Lambda_w(R_a f) = \int_N f(wna) \, dn \]

\[ = \int_N f(wa \cdot a^{-1}na) \, dn \]

\[ = \int_N f(waw^{-1} \cdot w \cdot a^{-1}na) \, dn \]

\[ = \delta^{1/2} \chi(a^{-1}) \int_N f(w \cdot a^{-1}na) \, dn \]

\[ = \delta^{-1/2}(a) \chi^{-1}(a) \cdot \delta(a) \int_N f(wn) \, dn \]

\[ = \delta^{1/2}(a) \chi^{-1}(a) \Lambda_w(f) \]

giving rise to a \( G \)-homomorphism

\[ T_w: \text{Ind}^\infty(\chi) \longrightarrow \text{Ind}^\infty(\chi^{-1}) \]
When is the integral

\[ \Lambda_w(f) = \int_N f(wn) \, dn = \int_{\mathbb{R}} f(wn_x) \, dx \quad (n_x = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix}) \]

defined? Since

\[ wn_x = \begin{bmatrix} 1/\sqrt{x^2 + 1} & \cdots \\ \sqrt{x^2 + 1} & \sqrt{x^2 + 1} \end{bmatrix} \begin{bmatrix} x/\sqrt{x^2 + 1} \\ 1/\sqrt{x^2 + 1} \end{bmatrix} \begin{bmatrix} 1/\sqrt{x^2 + 1} & -1/\sqrt{x^2 + 1} \\ 1/\sqrt{x^2 + 1} & x/\sqrt{x^2 + 1} \end{bmatrix} \]

\[ f(wn_x) = (x^2 + 1)^{(s+1)/2} f(kx) \]

and

\[ \Lambda_w(f) = \int_{\mathbb{R}} (x^2 + 1)^{(s+1)/2} f(kx) \, dx \]

Since \((x^2 + 1)^{(s+1)/2} \sim 1/x^{s+1}\) this converges and is holomorphic for \(\text{Re}(s) > 0\).
Explicitly

\[
\Lambda_w(\varphi_0) = \int_{\mathbb{R}} (x^2 + 1)^{-\frac{s+1}{2}} \, dx = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}
\]

since \( \varphi_0(kx) = 1 \). This continues meromorphically to all of \( \mathbb{C} \).

Similarly

\[
\Lambda_w(\varphi_1) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)}
\]

Since \( x_{\pm} \) commutes with \( T_w \) and \( x_{\pm} \cdot \varepsilon_n = (s + 1 \pm n) \varepsilon_n \) we see that \( \Lambda_w \) is meromorphic on all of \( \text{Ind}(\chi) \).

In fact it is meromorphic on all of \( \text{Ind}^\infty(\chi) \), but we’ll postpone checking that.
9. Characters
If \((\pi, V)\) is any smooth representation of \(G\) and \(f\) lies in \(C_c^\infty(G)\) then

\[
[\pi(f)](v) = \int_G f(g)\pi(g)v \, dg
\]

defines \(V\) as a module over \(C_c^\infty(G)\). This is an element of the vector space of continuous linear maps from \(V\) to itself.

If \(V\) has finite dimension then \(\text{Hom}_C(V, V) = \hat{V} \otimes V\), and \(\pi(f)\) would be an element of this tensor product. One can introduce a topological tensor product that allows us to make the same assertion for a large class of smooth representations, but here I’ll look at the case of \(V = \text{Ind}^\infty(\chi| P, G)\). I shall define \(\pi(f)\) as an element of

\[
\text{Ind}(\chi^{-1} \otimes \chi | P \times P, G \times G),
\]

which is in fact a topological tensor product of \(\hat{V} \otimes V\) when \(V\) is \(\text{Ind}^\infty(\chi)\).
For any $f$ in $C_c^\infty(G)$ define

$$f_\chi(g, h) = \int_A \chi \delta_P^{-1/2}(a) \, da \int_N f(h^{-1}nag) \, dn,$$

a function on $G \times G$.

**Proposition.** The function $f_\chi(g, h)$ lies in

$$\text{Ind}^\infty(\chi^{-1} \otimes \chi \mid P \times P, G \times G)$$
For example

\[ f_\chi(n^*g, h) = \int_A \chi \delta_P^{-1/2}(a) \, da \int_N f(h^{-1}na \cdot n^*g) \, dn \]

\[ = \int_A \chi \delta_P^{-1/2}(a) \, da \int_N f(h^{-1}n \cdot an^*a^{-1} \cdot ag) \, dn \]

\[ = \int_A \chi \delta_P^{-1/2}(a) \, da \int_N f(h^{-1}nag) \, dn \]

\[ = f_\chi(g, h) \]

and

\[ f_\chi(a^*_g, h) = \int_A \chi \delta_P^{-1/2}(a) \, da \int_N f(h^{-1}na \cdot a^*_g) \, dn \]

\[ = \int_A \chi \delta_P^{-1/2}(ba^*_1) \, db \int_N f(h^{-1}n \cdot bg) \, dn \]

\[ = \chi^{-1} \delta^{1/2}(a_*) \, f_\chi(g, h) \]
If $F$ lies in $\text{Ind}^\infty(\chi^{-1} \otimes \chi | P \times P, G \times G)$ and $\varphi$ in $\text{Ind}^\infty(\chi)$ then for each fixed $h$ in $G$ the product $F(g, h) \cdot \varphi(g)$ lies in $\Omega^\infty(P \backslash G)$, and hence the integral

$$\int_{P \backslash G} F(x, h)\varphi(x) \, dx = [F(\varphi)](h)$$

is defined. The map $\varphi \mapsto F(\varphi)$ is an endomorphism of $\text{Ind}^\infty(\chi)$.

If $V$ were finite-dimensional then for any $f$ in $\hat{V} \otimes V$ its trace when considered as an endomorphism of $V$ would be the image of $f$ under the canonical pairing

$$\hat{v} \otimes v \longmapsto \langle \hat{v}, v \rangle$$

This remains valid here. There is a canonical $G \times G$-covariant map from $\text{Ind}^\infty(\chi^{-1} \otimes \chi | P \times P, G \times G)$ to $\Omega^\infty(P \times P \backslash G \times G)$ and thence to $\mathbb{C}$ and the trace of $F$ is its image in $\mathbb{C}$. 
We can do things more concretely.

\[
R_f \varphi(g) = \int_G f(x) \varphi(gx) \, dx
\]

\[
= \int_G f(g^{-1}y) \varphi(y) \, dy
\]

\[
= \int_K dk \int_A \delta_P^{-1}(a) \, da \int_N \varphi(nak) f(g^{-1}nak) \, dn
\]

\[
= \int_K \varphi(k) \, dk \int_A \sigma(a) \delta_P^{-1/2}(a) \, da \int_N f(g^{-1}nak) \, dn.
\]

The trace of \( R_f \) on \( \text{Ind}^\infty(\chi) \) is therefore

\[
\int_A \chi \delta^{-1/2}(a) \, da \int_N \overline{f}(na) \, dn \quad \text{where} \quad \overline{f}(an) = \int_K f(kank^{-1}) \, dk
\]
The result we eventually want is this:

**Theorem.** The trace of $R_f$ on $\text{Ind}^\infty(\chi)$ is

$$\int_G f(g) \Theta \chi(g) \, dg$$

where

$$\Theta \chi(g) = \frac{\chi(x) + \chi^{-1}(x)}{|x - x^{-1}|}$$

if $g$ is conjugate to $a_x$ and 0 otherwise.

The point here is that the character of $\text{Ind}^\infty(\chi)$ is originally defined as a distribution, but it is in fact a distribution defined by the locally summable function $\Theta \chi$. 
We want to show that
\[ \int_A \chi \delta^{-1/2}(a) \, da \int_N \overline{f}(na) \, dn \]
is the same as
\[ \int_{G_A} f(g) \Theta \chi(g) \, dg \]
where \( \Theta \chi(g) = \frac{\chi(x) + \chi^{-1}(x)}{|x - x^{-1}|} \) if \( g \) is conjugate to \( a_x \).

We can write the first as
\[ \int_A \chi(a) \overline{f}_P(a) \, da \text{ where } f_P(a) = \delta^{-1/2}(a) \int_N f(na) \, dn \]
Because $\Theta$ is conjugation-invariant we can write the other integral as

$$\int_G f(g)\Theta(g)\,dg = \frac{1}{2} \int_A |\Delta(a)| \Theta(a)\,da \int_{G/A} f(xax^{-1})\,dx \quad (\text{Weyl})$$

$$= \frac{1}{2} \int_A |\Delta(a)| \frac{\chi(a) + \chi^{-1}(a)}{|\Delta(a)|^{1/2}}\,da \int_{G/A} f(xax^{-1})\,dx$$

$$= \frac{1}{2} \int_A |\Delta(a)|^{1/2} (\chi(a) + \chi^{-1}(a))\,da \int_{G/A} f(xax^{-1})\,dx$$

$$= \int_A \chi(a) |\Delta(a)|^{1/2} \,da \int_{G/A} f(xax^{-1})\,dx.$$

Here $\Delta(ax) = |x - x^{-1}|$. 
We want to show that

\[ \int_A \chi(a) \bar{f}_P(a) \, da = \int_A \chi(a) |\Delta(a)|^{1/2} \, da \int_{G/A} f(xax^{-1}) \, dx \]

i.e.

\[ \delta^{-1/2}(a) \int_N \, dn \int_K f(knak^{-1}) \, dk = |\Delta(a)|^{1/2} \int_{G/A} f(xax^{-1}) \, dx \]

This depends on a lemma of Harish-Chandra’s—*for any* \( a_x \) *in* \( A \)
*with* \( x^2 \neq 1 \) *the transformation* \( n \mapsto n \cdot ana^{-1} \) *is bijective with modulus* \( |\text{det Ad}_n(a) - 1| = |x^2 - 1| \).

You’ll need to know that \( |x^2 - 1| = |x||x - x^{-1}| = \delta^{1/2}(a_x) \Delta(a) \).
The End