COHEN-MACAULAY RINGS

In this hour we will talk about, or build up to talking about, Cohen-Macaulay rings. This is a class of rings that is closed under the operations of localization, completion, adjoining polynomial and power series variables, and taking certain quotients. First, we need to give some definitions.

Definition: Let $A$ be a ring and $M$ an $A$-module. An element $a \in A$ is said to be $M$-regular if $ax \neq 0$ for all $0 \neq x \in M$. In other words, $a$ is not a zero divisor on $M$.

Example: $M = A = k[x]$, where $k$ is a field. Then $x$ is regular on $A$.

Definition: A sequence $a_1, \ldots, a_r$ of elements of $A$ is an $M$-sequence (or an $M$-regular sequence) if the following two conditions hold:

1. $a_1$ is $M$-regular, $a_2$ is $M/a_1M$-regular, etc.
2. $M/\sum a_i M \neq 0$.

REMARK: If $a_1, a_2, \ldots, a_r$ is an $M$-sequence, then so is $a_1^{t_1}, \ldots, a_r^{t_r}$, for any positive integers $t_i$. However, just because $a_1, \ldots, a_r$ is an $M$-sequence does not mean that a permutation of $a_1, \ldots, a_r$ is an $M$-sequence. In order for any permutation of the sequence to be an $M$-regular sequence, we would need the ring to be Noetherian local and the module to be finite.

Example: The classical example of a regular sequence is $x_1, \ldots, x_r$ in the polynomial ring $A[x_1, \ldots, x_r]$.

Non-Example: Let $A = k[x, y, z]$, where $k$ is a field. Show that $x, y(1-x), z(1-x)$ is an $A$-sequence, but $y(1-x), z(1-x), x$ is NOT.

Note that $z(1-x)$ is not regular on $A/(y(1-x))$ since $z(1-x)y = zy - xyz = zy - zy$ since $y = yx$ in $A/(y(1-x))$.

Let $A$ be a Noetherian ring and $M$ a finite $A$-module. If $x = x_1, \ldots, x_n$ is an $M$-sequence, then the chain $(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \ldots, x_n)$ ascends strictly. Therefore, an $M$-sequence can be extended to a maximal such sequence, since $A$ is Noetherian, and hence the chain must terminate. An $M$-sequence $x_1, \ldots, x_n$ is maximal if $x_1, \ldots, x_{n+1}$ is NOT an $M$-sequence.
for any $x_{n+1} \in R$.

**REMARK:** All maximal $M$-sequences have the same length if $M$ is finite (and $A$ is Noetherian).

Let $A$ be a local ring.

**Definition:** We call this length the **depth** of $M$. If we are talking about $M$-sequences in a non-maximal ideal $I$ of $A$, then we use the notation $\text{depth}(I, M)$.

Another, more technical definition of depth, is $\inf\{i: \text{Ext}^i_A(k, M) \neq 0\}$, where $k$ is the residue field of $A$. Likewise, $\text{depth}(I, M) = \inf\{i: \text{Ext}^i_A(A/I, M) \neq 0\}$.

**Definition:** Let $(A, m, k)$ be a Noetherian local ring and $M$ a finite $A$-module. $M$ is called **Cohen-Macaulay (CM)** if $M \neq 0$ and $\text{depth} M = \dim M$. If $A$ is itself a Cohen-Macaulay module, we say that $A$ is a **Cohen-Macaulay ring**.

**QUESTION:** What happens if $A$ is not local?

**Definition:** A Noetherian ring $A$ is said to be a CM ring if $A_m$ is a CM local ring for every maximal ideal $m$ of $A$.

**Example:** The rings $k[X_1, \ldots, X_n], k[[X_1, \ldots, X_n]]$, $k[X, Y, Z]/(XY - Z)$, and $k[X, Y, Z, W]/(XY - ZW)$ are all Cohen-Macaulay, for example.

**Non-Example:** The ring $A = k[[X^4, X^3Y, XY^3, Y^4]] \subset k[X, Y]$ is NOT Cohen-Macaulay.

Note that $A$ has dimension 2 since $\{X^4, Y^4\}$ is an s.o.p.. In particular, $(X^3Y)^4 = X^{12}Y^4 \in (X^4, Y^4)$; likewise for $XY^3$. Thus, $m \subset (X^4, Y^4)$. We’ll show that $Y^4$ is not regular on $A/(X^4)$. $Y^4(X^6Y^2) = X^6Y^6 = X^4(X^2Y^6) \in (X^4)$, but $X^6Y^2 \notin (X^4)$ since $X^6Y^2 = X^4(X^2Y^2)$, and $X^3Y^2 \notin A$. Thus, we have found an s.o.p. that is not $A$-regular.

In order to show that the above example was not Cohen-Macaulay, we used
the following result.

**THEOREM:** Let \((A, \mathfrak{m})\) be a CM local ring.

(a) For a proper ideal \(I\) of \(A\) we have \(\text{ht}\ I = \text{depth}(I, A)\) and \(\text{ht}\ I + \dim\ A/I = \dim\ A\).

(b) For any sequence \(a_1, \ldots, a_r \in \mathfrak{m}\) the following four conditions are equivalent:

(i) \(a_1, \ldots, a_r\) is an \(A\)-sequence
(ii) \(\text{ht}(a_1, \ldots, a_i) = i\) for \(1 \leq i \leq r\)
(iii) \(\text{ht}(a_1, \ldots, a_r) = r\)
(iv) \(a_1, \ldots, a_r\) is part of a system of parameters of \(A\)

**THEOREM:** Let \((A, \mathfrak{m})\) be a Noetherian local ring and \(M\) a finite \(A\)-module.

(a) If \(a_1, \ldots, a_r \in \mathfrak{m}\) is an \(M\)-sequence and we set \(M' = M/(a_1, \ldots, a_r)M\), then \(M\) is a CM module \iff \(M'\) is a CM module.

(b) If \(M\) is a CM module then \(M_p\) is a CM module over \(A_p\) for every \(p \in \text{Spec}(A)\), and if \(M_p \neq 0\), then \(\text{depth}(p, M) = \text{depth}_{A_p} M_p\).

(c) Let \(\hat{A}\) be the \(\mathfrak{m}\)-adic completion of \(A\). Then (i) \(\text{depth}\ \hat{A} = \text{depth} A = \text{depth} \hat{A}\) and (ii) \(A\) is CM \iff \(\hat{A}\) is CM.

(d) If \(A\) is CM, then so are \(A[X]\) and \(A[[X]]\).

Sometimes one only needs a ring or module to be Cohen-Macaulay in “low dimension”. This concept is called the Serre condition \(S_i\); for \(i \geq 0\).

**Definition:** The condition \(S_i\) on \(A\) means that \(\text{depth}\ A_p \geq \min\{\text{ht}\ p, i\}\) for all \(p \in \text{Spec}(A)\).

**REMARK 1:** Of course, \(S_0\) always holds; \(S_1\) says that all the associated primes of \(A\) are minimal, that is \(A\) does not have embedded associated primes; for an integral domain, \(S_2\) is equivalent to the condition that every prime divisor of a non-zero principal ideal has height 1

**REMARK 2:** \((S_i)\) for all \(i \geq 0\) is just the definition for CM ring.

**REMARK 3:** If \(S_i\) holds on \(A\), then \(S_j\) also holds, for all \(j < i\).
**Definition:** To say that a finite module $M$ satisfies $S_i$ means that depth $M_p \geq \min\{\dim M_p, i\}$, for all $p \in \text{Spec}(A)$.

**Definition:** We can also define the **Serre condition** $R_i$, for $i \geq 0$. A Noetherian ring $A$ satisfies $R_i$ if $A_p$ is a regular local ring for all prime ideals $p$ in $A$ of height less than or equal to $i$.

**REMARK 1:** If $R_i$ holds on $A$, then $R_j$ also holds, for all $j < i$, since $A_p \cong (A_q)_p A_q$, for $p \subset q$.

**REMARK 2:** Recall that the a Noetherian ring $A$ is normal if and only if the conditions $S_2$ and $R_1$ are satisfied.

Recall that, in general, depth $M \leq \dim M \leq \dim A$, for a finite $A$-module $M$.

**Definition:** A **small CM module** is a finitely-generated module $M$ such that depth $M = \dim R$. We can also define a **big CM module**. Let $x_1, \ldots, x_d$ be a system of parameters. If $M$ is a (not necessarily finitely-generated) $A$-module such that $(x_1, \ldots, x_d)M \neq M$ and $x_1, \ldots, x_d$ is an $M$-regular sequence, then $M$ is called a big CM module.

**SMALL COHEN-MACAULAY CONJECTURE:** If $R$ is a complete local Noetherian ring, then $R$ has a small CM module.

**BIG COHEN-MACAULAY CONJECTURE:** If $R$ is a local Noetherian ring, then $R$ has a big CM module.
INJECTIVE MODULES AND GORENSTEIN RINGS

This class of Gorenstein rings is also closed under localization, completion, and adjoining polynomial and power series variables. We start with some definitions.

**Definition:** An $A$-module $I$ is called **injective** if given any diagram of $A$-module homomorphisms:

$$
\begin{array}{c}
0 \\
\Big\downarrow^f \\
I
\end{array}
\xrightarrow{g}
\begin{array}{c}
N \\
\Big\downarrow^L
\end{array}
\xrightarrow{h}
\begin{array}{c}
L \\
I
\end{array}
$$

with top row exact, there exists an $A$-module homomorphism $h : L \to I$ such that $h \circ g = f$.

**Definition:** Let $A$ be a Noetherian ring and $M$ and $N$ (not necessarily finitely-generated) $A$-modules. Then $M \hookrightarrow N$ is called **essential** (i.e., $N$ is an **essential extension** of $M$ if $N_0 \cap M \neq 0$ for every non-zero submodule $N_0$ of $N$.

**Definition:** An **injective hull** of $M$ is an injective module $E \supseteq M$ such that $M \hookrightarrow E$ is essential. The notation is $E(M)$.

**THEOREM:** Every injective module over a Noetherian ring is a direct sum of indecomposable injective modules.

**PROOF**

Say that a family $\mathcal{F} = \{E_\lambda\}$ of indecomposable (meaning the $E_\lambda$ can not be written as the direct sum of two submodules) injective submodules of $M$ is **free** if the sum in $M$ of the $E_\lambda$’s is direct; i.e., if, for any finite number $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_n}$ of them, $E_{\lambda_1} \cap (E_{\lambda_2} + \cdots + E_{\lambda_n}) = 0$. Let $\mathfrak{M}$ be the set of all free families $\mathcal{F}$, ordered by inclusion. By Zorn’s Lemma, $\mathfrak{M}$ has a maximal element, say $\mathcal{F}_0$. Write $N = \sum_{E \in \mathcal{F}_0} E$. Then $N$ is injective, since any direct sum of injective modules is injective. Since an injective submodule is always a direct summand, we have $M = N \oplus N'$. If $N' \neq 0$, then since it’s a direct summand of the injective module $M$ it must be injective itself. Let $p \in \text{Ass}(N')$. Then we have the following diagram:
where $L \to N'$ is an injection since the other two maps are injections. Thus, $N'$ contains a direct summand $E'$ isomorphic to $E(A/p)$, which is indecomposable. Since $M = N \oplus N'$, this means that $E' \cap N = 0$; i.e., $\mathfrak{g}_0 \cup \{E'\}$ is a free family, contradicting the maximality of $\mathfrak{g}_0$. Hence $N' = 0$, and $M = N$.

Definition: An $A$-module $M$ has **injective dimension** $\leq n$ (inj dim $M \leq n$) if there is an injective resolution

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$$

If no such finite resolution exists, then inj dim $(M)$ is defined to be $\infty$.

Definition: A Noetherian local ring $A$ is a **Gorenstein ring** if any of the following equivalent conditions hold:

1. inj dim $(A) < \infty$
2. inj dim $(A) = n$
3. $\text{Ext}^i_A(k, A) = 0$ for $i \neq n$ and $\cong k$ for $i = n$
4. $\text{Ext}^i_A(k, A) = 0$ for some $i > n$
5. $\text{Ext}^i_A(k, A) = 0$ for $i < n$ and $\cong k$ for $i = n$
6. $A$ is a CM ring and $\text{Ext}^n_A(k, A) \cong k$

Definition: The **type** of a non-zero finite $A$-module $M$ of depth $t$ is $r(M) = t$.
\[ \dim_k \Ext_A^i(k, M). \]

**Example:** The rings \( k[X_1, \ldots, X_n], k[[X_1, \ldots, X_n]], k[X, Y, Z]/(XY - Z), \) and \( k[X, Y, Z, W]/(XY - ZW) \) are Gorenstein rings.

**Non-Example:** (Gor \( \subset \) CM) Let \( A = k[X, Y]/(X^2, XY, Y^2) \), which has dimension 0, and hence is a CM ring. The depth of \( A \) is also zero, so the type of \( A \) is \( \dim_k \Hom_A(k, A) \). Now \( \text{Soc}(A) = (0 : m)_A \cong \Hom_A(k, A) = m \) in this case. Thus, \( \dim_k \Hom_A(k, A) = \dim_k m/m^2 = 2 \). Therefore, since type of \( A \) is greater than 1, \( A \) can not be Gorenstein.

**Definition:** A Noetherian ring is a **Gorenstein ring** if its localization at every maximal ideal is a Gorenstein local ring.

**THEOREM:** Let \( A \) be a Noetherian ring.

(a) Suppose \( A \) is Gorenstein. Then for every multiplicatively closed set \( S \) in \( A \) the localized ring \( A_S \) is also Gorenstein. In particular, \( A_p \) is Gorenstein for every \( p \in \text{Spec}(A) \).

(b) Suppose \( x \) is an \( A \)-regular sequence. If \( A \) is Gorenstein, then so is \( A/(x) \).

(c) Suppose \( A \) is local. Then \( A \) is Gorenstein if and only if its completion \( \hat{A} \) is Gorenstein.

(d) If \( A \) is Gorenstein, then so are \( A[X] \) and \( A[[X]] \).

**REMARK:** One way to check if a ring is Gorenstein is to first check if it is Cohen-Macaulay. If so, then find a system of parameters, kill it, and then compute the Socle; determine how many linearly independent elements are in the Socle. This is the type.