1. Find the symmetric equations of the line through the point (3,2,1) and perpendicular to the plane $7x - 3y + z = 14$.

**Solution.** The vector $\mathbf{V} = 7\mathbf{I} - 3\mathbf{J} + \mathbf{K}$ is orthogonal to the given plane, so points in the direction of the line. If we let $\mathbf{X}_0 = 3\mathbf{I} + 2\mathbf{J} + \mathbf{K}$, then the condition for $\mathbf{X}$ to be the vector to a point on the line is $\mathbf{X} - \mathbf{X}_0$ is collinear with $\mathbf{V}$, which gives us the symmetric equations

$$\frac{x - 3}{7} = \frac{y - 2}{-3} = \frac{z - 1}{1}.$$

2. Find the equation of the plane through the points $(0,-1,1)$, $(1,0,1)$ and $(1,2,2)$.

**Solution.** The vectors from the first point (call it $\mathbf{X}_0$) to the second and third points are $\mathbf{U} = \mathbf{I} + \mathbf{J}$, $\mathbf{V} = \mathbf{I} + 3\mathbf{J} + \mathbf{K}$. Since $\mathbf{U}, \mathbf{V}$ lie on the plane $\mathbf{U} \times \mathbf{V}$ is normal to the plane. We calculate $\mathbf{N} = \mathbf{U} \times \mathbf{V} = \mathbf{I} - \mathbf{J} + 2\mathbf{K}$. Thus the equation of the plane is

$$\mathbf{X} \cdot \mathbf{N} = \mathbf{X}_0 \cdot \mathbf{N}, \quad \text{or} \quad x - y + 2z = 3.$$

3. A particle moves through the plane as a function of time: $\mathbf{X}(t) = t^2\mathbf{I} + 2t^3\mathbf{J}$. Find the unit tangent and normal vectors, and the tangential and normal components of the acceleration.

**Solution.** $\mathbf{V} = 2t\mathbf{I} + 6t^2\mathbf{J}$, $\mathbf{A} = 2\mathbf{I} + 12t\mathbf{J}$. Thus $ds/dt = 2t\sqrt{1 + 9t^2}$ and

$$\mathbf{T} = \frac{\mathbf{I} + 3t\mathbf{J}}{\sqrt{1 + 9t^2}}, \quad \mathbf{N} = \frac{-3\mathbf{I} + \mathbf{J}}{\sqrt{1 + 9t^2}}.$$

Then

$$a_T = \mathbf{A} \cdot \mathbf{T} = \frac{2 + 36t^2}{\sqrt{1 + 9t^2}}, \quad a_N = \mathbf{A} \cdot \mathbf{N} = \frac{6t}{\sqrt{1 + 9t^2}}.$$

4. A particle moves through space as a function of time:

$$\mathbf{X}(t) = \cos t\mathbf{I} + t\sin t\mathbf{J} + t\mathbf{K}.$$

For this motion, find $\mathbf{T}, \mathbf{N}$, the the tangential and normal components of the acceleration, and the curvature at time $t = 3\pi/2$.

**Solution.** $\mathbf{V} = -\sin t\mathbf{I} + (\sin t + t\cos t)\mathbf{J} + \mathbf{K}$, $\mathbf{A} = -\cos t\mathbf{I} + (2\cos t - t\sin t)\mathbf{J}$. Evaluate at $t = 3\pi/2$:

$$\mathbf{V} = \mathbf{I} - \mathbf{J} + \mathbf{K}, \quad \mathbf{A} = \frac{3\pi}{2}\mathbf{J}.$$
Then \[ a_T = A \cdot T = A \cdot \frac{V}{|V|} = -\frac{\pi}{2} \sqrt{3} , \quad a_N N = A - a_T T = \frac{\pi}{2} (I + 2J + K) \]

so \[ a_N = \frac{\pi}{2} \sqrt{6} , \quad N = \frac{I + 2J + K}{\sqrt{6}} . \]

5. The particle of problem 3 moves in opposition to the force field \( F(x, y, z) = xI - yJ - K \). How much work is required to move the particle from \((1,0,0)\) to \((1,0,2\pi)\)?

**Solution.** The curve is parametrized by \( X(t) = \cos t I + t \sin t J + t K, \) \( 0 \leq t \leq 2\pi \), so along the curve \( F = \cos t I - t \sin t J - K \), \( dX = (-\sin t I + (\sin t + t \cos t)J + K)dt \).

Thus the work is \[ \int_C F \cdot dX = \int_0^{2\pi} (-\cos t \sin t - t \sin t (\sin t + t \cos t) - 1)dt = -2\pi . \]

This calculation can be avoided by noticing that \( F \) is a gradient field: \( F = \nabla f \), with \( f(x, y, z) = (x^2 - y^2)/2 - z \). Thus \[ \int_C F \cdot dX = f(1,0,2\pi) - f(1,0,0) = -2\pi . \]

6. Find the critical points of

\[ f(x, y) = 3xy + \frac{1}{x} - \ln y \]

in the first quadrant. Classify as local maximum or minimum or saddle point.

**Solution.** \( \nabla f = (3y - 1/x^2)I + (3x - 1/y)J \). We solve \( \nabla f = 0 \) : \( 3y = x^{-2} \), \( 3x = y^{-1} \) give \( x^2 = x \), so \( x = 0 \) or \( x = 1 \). Since \( x = 0 \) is not in the first quadrant, the only critical point is \((1,1/3)\). At this point \( f_{xx} = -2x^{-3} = -2 , \quad f_{yy} = y^{-2} = 9 \), and \( f_{xy} = 3 \). Thus \( D = (-2)(9) - 9 = -27 \), and \((1,1/3)\) is a saddle point.

7. The temperature distribution on the surface \( x^2 + y^2 + z^2 = 1 \) is given by \( T(x, y, z) = xz + yz \). Find the hottest spot.

**Solution.** In the language of Lagrange multipliers, the objective function is \( T(x, y, z) = xz + yz \), and the constraint is \( g(x, y, z) = x^2 + y^2 + z^2 = 1 \). The gradients are \( \nabla T = zI + zJ + (x+y)K \), \( \nabla g = 2(xI + yJ + zK) \). The Lagrange equations are

\[ z = \lambda x , \quad z = \lambda y , \quad x + y = \lambda z , \quad x^2 + y^2 + z^2 = 1 . \]
Now, at $z = 0$, we have $T = 0$, so no such point is the hottest spot. The first equations therefore give us $x = y$. Replacing $y$ by $x$ we now have

$$z = \lambda x, \quad 2x = \lambda z, \quad 2x^2 + z^2 = 1.$$ 

By the first two equations, $\lambda^2 = 2$, and the last equation then gives us $2x^2 + 2x^2 = 1$, so $x^2 = 1/4$. These then are the critical points:

$$x = y = \pm \frac{1}{2}, \quad z = \pm \frac{1}{\sqrt{2}}.$$ 

$T$ takes its maximum at $(1/2, 1/2, 1/\sqrt{2})$, and its negative.

8. What is the equation of the tangent plane to the surface $z^2 - 3x^2 - 5y^2 = 1$ at the point $(1,1,3)$?

**Solution.** Take the differential of the defining equation of the surface: $2zdz - 6xdx + 10ydy = 0$. Substitute the coordinates of the point $(1,1,3)$: $6dz - 6dx + 10dy = 0$. This is the equation of the tangent plane, with the differentials replaced by the increments:

$$6(z - 3) - 6(x - 1) + 10(y - 1), \quad \text{or} \quad -6x + 10y + 6z = 22.$$ 

9. Consider the surface $\Sigma$

$$f(x,y) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1.$$ 

a) At what points on $\Sigma$ is the tangent plane parallel to the plane $2x + y - z = 1$?

**Solution.** The normal to the plane is $N = 2I + J - K$. The surface is given as a level set of the function $f$, so its normal is

$$\nabla f(x,y) = \frac{x}{2}I + 2yJ + \frac{2z}{9}K.$$ 

The places on $\Sigma$ where the tangent plane is parallel to the given plane are those values of $(x,y)$ where $\nabla f(x,y)$ is colinear with $N$. These are the solutions of the system of equations:

$$x = 4\lambda, \quad y = \frac{\lambda}{2}, \quad z = -\frac{9\lambda}{2}, \quad \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1.$$ 

Putting the expressions in $\lambda$ given by the first three equations into the fourth, we can solve for $\lambda$, getting

$$\lambda = \pm \frac{2}{\sqrt{26}}.$$ 

Thus there are two solutions to the problem:

$$\left(\frac{8}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{9}{\sqrt{26}}\right), \quad \left(-\frac{8}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, \frac{9}{\sqrt{26}}\right).$$
b) What constrained optimization problem is solved by part a)?

Solution. Find the maximum and minimum of \(2x + y - z\) on the surface \(\Sigma\).

10. Find the volume of the tetrahedron in the first octant bounded by the plane

\[
\frac{x}{5} + \frac{y}{3} + \frac{z}{2} = 1.
\]

Solution. Draw the picture to see that we can represent the region by the inequalities

\[
0 \leq x \leq 5, \quad 0 \leq y \leq 3(1 - \frac{x}{5}), \quad 0 \leq z \leq 2(1 - \frac{x}{5} - \frac{y}{3}).
\]

So the volume is given by the iterated integral

\[
\int_0^5 \int_0^{3(1 - \frac{x}{5})} \int_0^{2(1 - \frac{x}{5} - \frac{y}{3})} dz \, dy \, dx = \frac{15}{2}.
\]

11. a) Find the volume of the solid in the first quadrant which lies over the triangle with vertices \((0,0), (1,0), (1,3)\) and under the plane \(z = 2x + 3y + 1\).

Solution. The solid is that under the given plane and lying over the triangle \(T: 0 \leq x \leq 1, \ 0 \leq y \leq 3x\). Its volume is

\[
Volume = \int \int_T z \, dy \, dx = \int_0^1 \int_0^{3x} (2x + 3y + 1) \, dy \, dx.
\]

The inner integral is

\[
2xy + \frac{3y^2}{2} \bigg|_0^{3x} = \frac{21}{2} x^2 + 3x.
\]

Thus

\[
Volume = \int_0^1 \left( \frac{21}{2} x^2 + 3x \right) \, dx = \frac{21}{6} + \frac{3}{2} = 5.
\]

b) Find the area of that segment of the plane.

Solution. The element of surface area is \(dS = \sqrt{1 + z_x^2 + z_y^2} \, dy \, dx = \sqrt{1 + 2^2 + 3^2} \, dy \, dx = \sqrt{14} \, dy \, dx\). Thus the area of the triangle on the surface is \(\sqrt{14}\) times the area of the triangle, so is \(3\sqrt{14}/2\).

12. Find the area of the region in the first quadrant bounded by the parabolas

\[
y^2 - x = 1, \quad y^2 - x = 0, \quad y^2 + x = 5, \quad y^2 + x = 4.
\]
Solution. Make the change of variable: \( u = y^2 - x, \ v = y^2 + x \). Then in the \( uv \)-plane the region is described by \( 0 \leq u \leq 1, 4 \leq v \leq 5 \). We need to calculate the Jacobian; for that we solve for \( x \) and \( y \) in terms of \( u \) and \( v \):

\[
x = \frac{v - u}{2} \quad y = \frac{(u + v)^{1/2}}{\sqrt{2}}.
\]

Then

\[
\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{(u+v)^{-1/2}}{2\sqrt{2}} & \frac{(u+v)^{-1/2}}{2\sqrt{2}}
\end{pmatrix} = \frac{-(u + v)^{-1/2}}{2\sqrt{2}}.
\]

The area then is the integral

\[
Area = \frac{1}{2\sqrt{2}} \int_0^1 \int_4^5 (u + v)^{-1/2} dvdu.
\]

The inner integral is \( 2[(u + 5)^{1/2} - (u + 4)^{1/2}] \), so

\[
Area = \frac{1}{\sqrt{2}} \int_0^1 [(u + 5)^{1/2} - (u + 4)^{1/2}] \, du = \frac{\sqrt{2}}{3}(6^{3/2} + 4^{3/2} - 5^{3/2})
\]

13. Find the mass of a lamina over the domain in the plane \( D : 0 \leq y \leq x(1 - x) \), if the density function is \( \delta(x, y) = 1 + x + y \).

Solution.

\[
Mass = \int \int_D \delta \, dA = \int_0^{x(1-x)} \int_0^{1-x} (1 + x + y) \, dydx.
\]

The inner integral is

\[
y + xy + \frac{y^2}{2} \bigg|_0^{x(1-x)} = \frac{x^4}{4} - 2x^3 + \frac{x^2}{2} + x.
\]

Thus

\[
Mass = \frac{1}{10} - \frac{2}{4} + \frac{1}{6} + \frac{1}{2} = \frac{4}{15}.
\]

14. Find the center of mass of the piece of the unit sphere in the first octant:

\[
x^2 + y^2 + z^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.
\]

Solution. The volume of the sphere of radius 1 is \( 4\pi/3 \); the piece we’re looking at is \((1/8)\)th of that so \( Mass = \pi/6 \). Now, using spherical coordinates

\[
Mom_{x=0} = \int \int \int_R x \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho \sin \phi \cos \theta \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.
\]

\[5\]
Thus the $x$-coordinate of the center of mass is

\[
\bar{x} = \frac{Mom_{x=0}}{\text{Mass}} = \frac{\pi}{16} = \frac{3}{8}.
\]

15. Let

\[f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.
\]

Find a) $\nabla f$, b) curl $\nabla f$, c) div $\nabla f$, d) $\nabla (\text{div} \nabla f)$.

**Solution.**

a) \[
\nabla f = \left( \frac{1}{y} - \frac{z}{x^2} \right) \mathbf{I} + \left( \frac{1}{z} - \frac{x}{y^2} \right) \mathbf{J} + \left( \frac{1}{x} - \frac{y}{z^2} \right) \mathbf{K}
\]

b) \[
\text{curl} \nabla f = 0
\]

c) \[
\text{div} \nabla f = \frac{2z}{x^3} + \frac{2x}{y^3} + \frac{2y}{z^3}
\]

d) \[
\nabla (\text{div} \nabla f) = \left( -\frac{6z}{x^4} + \frac{2}{y^3} \right) \mathbf{I} + \left( -\frac{6x}{y^4} + \frac{2}{z^3} \right) \mathbf{J} + \left( -\frac{6y}{z^4} + \frac{2}{x^3} \right) \mathbf{K}
\]

e) \[
\nabla \times \nabla (\text{div} \nabla f) = 0
\]

16. Let $\mathbf{F} = (y + 2xz) \mathbf{I} + (x + z^2 + 1) \mathbf{J} + (2yz + x^2) \mathbf{K}$. Find a function $f$ such that $\mathbf{F} = \nabla f$.

**Solution.** We want to solve the equations

\[
\frac{\partial f}{\partial x} = y + 2xz, \quad \frac{\partial f}{\partial y} = x + z^2 + 1, \quad \frac{\partial f}{\partial z} = 2yz + x^2.
\]

The general solution of the first equation is $f(x, y, z) = xy + x^2z + \phi(y, z)$. Substitute this in the second equation to get

\[
\frac{\partial f}{\partial y} = x + \frac{\partial \phi}{\partial y} = x + z^2 + 1,
\]
leading to the equation for $\phi$:
\[
\frac{\partial \phi}{\partial y} = z^2 + 1.
\]
This has the solution $\phi(y,z) = z^2y + y + \psi(z)$. This now gives this form for $f$:
\[
f(x,y,z) = xy + x^2z + z^2y + y + \psi(z).
\]
Substitute that in the last equation to get
\[
\frac{\partial f}{\partial z} = x^2 + 2zy + \psi'(z) = 2yz + x^2,
\]
so that we must have $\psi'(z) = 0$, or $\psi(z) = C$. Thus the answer is
\[
f(x,y,z) = xy + x^2z + z^2y + y + C.
\]
17. Let $C$ be the curve in space given parametrically by the equations
\[
x = t^2 - 3t + 5, \quad y = (t^3 - 2)^2, \quad z = t^4 + t^3 - t^2, \quad 0 \leq t \leq 1,
\]
and $\mathbf{F}$ the vector field
\[
\mathbf{F}(x,y,z) = x\mathbf{I} + z\mathbf{J} + y\mathbf{K}.
\]
What is $\int_C \mathbf{F} \cdot d\mathbf{X}$?

**Solution.** Before doing the hair-raising direct calculation, note that $\mathbf{F} = \nabla (x^2 + y^2 + z^2)/2$. Thus we need only evaluate this function at the endpoints, which are $(5,4,0)$ (for $t = 0$) and $(3,1,1)$ (for $t = 1$). Thus
\[
\int_C \mathbf{F} \cdot d\mathbf{X} = \frac{x^2 + y^2 + z^2}{2} \bigg|_{(5,4,0)}^{(3,1,1)} = 15.
\]

18. Let $C$ be the curve given in polar coordinates by $r = 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$. Calculate $\int_C x\,dy$.

**Solution.** Parametrize the curve by $x = (1 + \cos \theta) \cos \theta$, $y = (1 + \cos \theta) \sin \theta$, $0 \leq \theta \leq 2\pi$, so that
\[
\int_C x\,dy = \int_0^{2\pi} (1 + \cos \theta) \cos \theta \, d((1 + \cos \theta) \sin \theta)
\]
\[
= \int_0^{2\pi} (1 + \cos \theta) \cos \theta (-\sin^2 \theta + (1 + \cos \theta) \cos \theta) \, d\theta = \frac{3}{2} \pi.
\]
Instead of computing that awful integral, we could note that, by Green’s theorem, the desired integral is the area of the cardioid $D$ bounded by $C$, so
\[
\int_C x\,dy = \text{Area}(D) = \frac{1}{2} \int_0^{2\pi} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 \, d\theta
\]
\[
\frac{1}{2} \int_{0}^{2\pi} \left( 1 + 2 \cos \theta + \frac{1 + \cos(2\theta)}{2} \right) d\theta = \frac{3}{2} \pi.
\]

19. Let \( C \) be the part of the curve \( y = x^2(24 - x) \) which lies in the first quadrant. Consider it directed from the point (0,0) to the point (24,0). Calculate
\[
\int_{C} (y + 1)\,dx - x\,dy.
\]

**Solution.** We can parametrize this curve by \( y = 24x^2 - x^3, \ 0 \leq x \leq 24; \) in which case \( dy = (48x - 3x^2)\,dx \) and we get
\[
\int_{C} (y + 1)\,dx - x\,dy = \int_{0}^{24} (24x^2 - x^3 + 1)\,dx - x(48x - 3x^2)\,dx
\]
\[
= \int_{0}^{24} (-24x^2 - 4x^3 + 1)\,dx = -442344.
\]