In problems 1-4, find the limits.

1. \( \lim_{x \to 0} \frac{\cos x - 1}{x^2} \)
   \( \text{Answer.} \) \( = \lim_{x \to 0} \frac{-\sin x}{2x} = -\frac{1}{2} \)

2. \( \lim_{x \to \pi} \frac{(x - \pi)^3}{\sin x + x - \pi} \)
   \( \text{Answer.} \) \( = \lim_{x \to \pi} \frac{3(x - \pi)^2}{\cos x + 1} = \lim_{x \to \pi} \frac{6(x - \pi)}{-\sin x} = \lim_{x \to \pi} \frac{6}{-\cos x} = 6 \)

3. \( \lim_{x \to \infty} x^5 e^{-x} \)
   \( \text{Answer.} \) \( = \lim_{x \to \infty} \frac{x^5}{e^x} = 0, \)
   which converges to zero since the exponential grows faster than any polynomial.

4. \( \lim_{x \to \infty} \frac{\sqrt{1 + x^2} - x}{x} \)
   \( \text{Answer.} \) \( = \lim_{x \to \infty} \frac{1}{x^2} (1 + 1 - 1) = 0, \)
   since \( x^2 \to 0 \) as \( x \to \infty \). We arrived at the second formulation from the first by dividing both numerator and denominator by \( x \). Observe that, although l’Hôpital’s rule applies, it doesn’t get us anywhere.

In problems 5-7: Does the integral converge or diverge? If you can, find the value of the integral.

5. \( \int_0^\infty xe^{-x^2} \, dx = \frac{1}{2} \int_0^\infty e^{-u} \, du = \frac{1}{2}, \)
   using the substitution \( u = x^2, \) \( du = 2x \, dx \) and a known computation (see example 8.16).

6. \( \text{Answer.} \) \( \int_0^\infty \frac{x^2}{x^3 + 1} \, dx \) diverges, since
   \( \frac{x^2}{x^3 + 1} = \frac{1}{x + \frac{1}{x^2}} \geq \frac{1}{2x} \)
   for \( x \) sufficiently large, and our knowledge that \( \int_0^\infty dx/x \) diverges.
7. \[ \int_0^1 \frac{dx}{x^{9/10}} \]

**Answer.** \( \lim_{a \to 0} \int_a^1 \frac{dx}{x^{9/10}} = \lim_{a \to 0} \left. 10x^{1/10} \right|_a^1 = 10 \)

8. Does the sequence converge or diverge?

a) \( a_n = \frac{n^2}{n!} \)

**Answer.** \( a_n = \frac{n^2}{n!} = \frac{n^2}{n(n-1)(n-2)!} = \left( \frac{1}{1-\frac{2}{n}} \right) \frac{1}{(n-2)!} \to 0 \)

since the first factor converges to 1, while the second converges to 0.

b) \( b_n = \frac{\sqrt{n^4}}{(n+1)^2} \)

**Answer.** \( b_n = \frac{\sqrt{n!}}{(n+1)^2} = \sqrt{\frac{n!}{(n+1)^2}} \to \infty \)

because the expression under the square root sign goes to infinity (which we can show by an argument similar to that in part a).

c) \( c_n = \frac{n^3 - 50n + 1}{n^4 + 123n^3 + 1} \)

**Answer.** \( c_n = \frac{n^3 - 50n + 1}{n^4 + 123n^3 + 1} = \frac{1 - \frac{50}{n^3} + \frac{1}{n^4}}{n(1 + \frac{123}{n} + \frac{1}{n^3})} \to 0 \)

since every factor converges to 1 except that \( n \to \infty \).

9. Does the series converge or diverge?

a) \( \sum_{n=1}^{\infty} \frac{n^2}{n!} \)

**Answer.** This converges by the ratio test: \( \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} = \left( 1 + \frac{1}{n} \right)^2 \cdot \frac{1}{n+1} \to 0 \)

which is less than 1.

b) \( \sum_{n=1}^{\infty} \frac{\sqrt{n^4}}{(n+1)^2} \)

**Answer.** This diverges by 9b: the general term does not go to 0.

c) \( \sum_{n=20}^{\infty} \frac{n^3 - 50n + 1}{n^4 + 123n^3 + 1} \)

**Answer.** This diverges because \( \frac{n^3 - 50n + 1}{n^4 + 123n^3 + 1} = \frac{1 - \frac{50}{n^3} + \frac{1}{n^4}}{n(1 + \frac{123}{n} + \frac{1}{n^3})} > \frac{1}{2n} \)

eventually. By comparison with \( \sum(1/n) \) the series diverges.
10. Does the series converge or diverge?

a) \( \sum_{n=1}^{\infty} \frac{3n+1}{n^{3/2}} \) converges
by comparison with the series \( \sum(1/n^{3/2}) \):
\[
\frac{3n+1}{n^{3/2}} = \frac{3 + \frac{1}{n}}{n^{3/2}} < \frac{4}{n^{3/2}}
\]

b) \( \sum_{n=1}^{\infty} \frac{3^n n!}{(n+1)!5^n + 1} \) converges
by comparison with the geometric series:
\[
\frac{3^n n!}{(n+1)!5^n + 1} = \frac{1}{n+1} \left( \frac{3^n}{5^n + (\frac{1}{n+1})} \right) \leq \left( \frac{3}{5} \right)^n
\]

c) \( \sum_{n=1}^{\infty} \frac{(2n)!(n+1)}{(2n+1)!} \) diverges
since the general term does not converge to 0:
\[
\frac{(2n)!(n+1)}{(2n+1)!} = \frac{n+1}{2n+1} \rightarrow \frac{1}{2}
\]

d) \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}(3n+1)} \) converges
by comparison with the series \( \sum(1/n^{3/2}) \):
\[
\frac{1}{n^{1/2}(3n+1)} < \frac{1}{3n^{3/2}}
\]

11. Find the radius of convergence of the series:

a) \( \sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3} \)
Answer. We observe that this is the thrice differentiated geometric series, so \( R = 1 \). However we can use the ratio test for the coefficients:
\[
\frac{(n+1)n(n-1)}{n(n-1)(n-2)} = \frac{n+1}{n-2} \rightarrow 1
\]

b) \( \sum_{n=0}^{\infty} (2^n + 1)x^n \)
Answer. Write down the ratio of successive coefficients and divide numerator and denominator by \( 2^n \):
\[
\frac{2^{n+1} + 1}{2^n + 1} = \frac{2 + \frac{1}{2^n}}{1 + \frac{1}{2^n}} \rightarrow 2,
\]
so the radius of convergence is 1/2.

c) \( \sum_{n=1}^{\infty} \frac{3n^2 + 1}{n^4 + 1}(x + 1)^n \)
Answer. The coefficient looks like \( 3/n \) and so the series converges if \( |x + 1| < 1 \), and diverges outside this interval. Thus \( R = 1 \).
12. Find the Maclaurin series for $(1 + x)^{-3}$.

**Answer.** Starting with the geometric series, substitute $-x$ for $x$:

$$(1 + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Now, differentiate twice:

$$-(1 + x)^{-2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$2(1 + x)^{-3} = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2}$$

so

$$\frac{1}{(1 + x)^3} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n + 2)(n + 1) x^n$$

13. Find the Maclaurin series for $\int_0^x \arctan t \, dt$.

**Answer.** We start by substituting $-x^2$ for $x$ in the geometric series:

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Now integrate twice:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$

$$\int_0^x \arctan t \, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+2)(2n+1)},$$

14. Find the Maclaurin series for $x \ln(x + 1)$.

**Answer.** Once again start with the geometric series, with $-x$ for $x$

$$(1 + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Integrate and multiply by $x$:

$$\ln x x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n+1}.$$

15. Find the terms up to fourth order for the Maclaurin series for

$$\frac{e^x}{1 + x}$$

**Answer.** We write down the Maclaurin series for each of $e^x$, $1/(1 + x)$, explicitly, that is, term by term, up to the fourth order:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$
\[
\frac{1}{1-x} = 1 - x + x^2 - x^3 + x^4 + \cdots
\]

Now, we multiply these together as if they were polynomials, relegating all terms of order greater than 4 to the \( \cdots \):
\[
\frac{e^x}{1+x} = (1 + x + x^2 + \frac{x^3}{6} + \frac{x^4}{24} + \cdots)(1 - x + x^2 - x^3 + x^4 + \cdots)
\]
\[
= (1 - x + x^2 - x^3 + x^4) + (x - x^2 + x^3 - x^4) + (\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{2}) + (\frac{x^3}{6} - \frac{x^4}{6}) + \frac{x^4}{24} + \cdots
\]
where we have done the multiplication by successively multiplying the second series by the terms of the first. Now we collect terms;
\[
\frac{e^x}{1+x} = 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{9}{24}x^4 + \cdots
\]
(Why have all the terms in the first two parentheses, except 1, cancelled?)