Mathematics 1220-90 Summer, 2003, Final Examination Answers

1. Find the integral: \( \int \frac{\ln x}{x^2} \, dx \)

**Answer.** We integrate by parts to replace the term \( \ln x \) by a monomial. Make the substitution \( u = \ln x, \ dv = \frac{dx}{x^2}, \ du = \frac{dx}{x}, \ v = - \frac{1}{x} \). Then

\[
\int \frac{\ln x}{x^2} \, dx = - \frac{\ln x}{x} + \int \frac{dx}{x^2} = - \frac{\ln x}{x} - \frac{1}{x} + C.
\]

2. Integrate \( \int_1^4 (x^2 + 3x) \sqrt{x} \, dx \)

**Answer.** Just do the multiplication and integrate;

\[
\int_1^4 (x^{5/2} + 3x^{3/2}) \, dx = \frac{2}{7} x^{7/2} + \frac{3}{5} x^{5/2} \bigg|_1^4 = \frac{254}{7} + \frac{62}{5} = 73.486.
\]

3. Integrate: \( \int \frac{u^2 + 1}{u^2(u - 1)} \, du \)

**Answer.** We integrate by parts; that is, we find \( A, B, C \) such that

\[
\frac{u^2 + 1}{u^2(u - 1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u - 1}.
\]

Putting the right side over a common denominator and equating numerators gives \( u^2 + 1 = Au(u - 1) + B(u - 1) + Cu^2 \). Now evaluate at the roots:

At \( u = 0 \) : \( 1 = B(-1) \) so that \( B = -1 \),

At \( u = 1 \) : \( 1^2 + 1 = C \) so that \( C = 2 \).

Now we equate the coefficients of \( u^2 \): \( 1 = A + C \), so that \( A = -C + 1 = -1 \). This gives us

\[
\int \frac{u^2 + 1}{u^2(u - 1)} \, du = \int -\frac{1}{u} \, du + \int -\frac{1}{u^2} \, du + \int \frac{2}{u - 1} \, du = -\ln |u| + u^{-1} + 2 \ln |u - 1| + C.
\]

4. Four years ago I invested $10,000 in an account bearing continuously compounded interest. Today I have $13,500. Assuming that the same interest rate continues into the future, when will my account have $20,000?
Answer. We use the equation $P = P_0e^{rt}$. We are given: at $t = 0$, $P_0 = 10000$; at $t = 4$, $P = 13500$. Thus we can solve for the interest rate;

$$135 = 100e^{4r}, \quad \text{giving} \quad r = \frac{\ln(1.35)}{4} = .075 .$$

Now we want to know how long it takes $13500$ to grow to $20000$ at that rate;

$$200 = 135e^{.075t}, \quad \text{giving} \quad t = \frac{\ln(200/135)}{.075} = 5.241$$

more years.

5. Find the limit. Show your work.

a) Answer. $\lim_{x \to 4} \frac{\sin(\pi x)}{x^2 - 16} = \lim_{x \to 4} \frac{\pi \cos(\pi x)}{2x} = \frac{\pi}{8}$.

b) Answer. $\lim_{x \to 0} \frac{e^x - 1 - x}{2x^2} = \lim_{x \to 0} \frac{e^x - 1}{4x} = \lim_{x \to 0} \frac{e^x}{4} = \frac{1}{4}$.

c) Answer. $\lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{3x^2}{e^x} = \lim_{x \to \infty} \frac{6x}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0$.

6. Do the integrals converge? If so, evaluate:

a) Answer. $\int_1^\infty \frac{dx}{1 + x^2} = \lim_{A \to \infty} \int_1^A \frac{dx}{1 + x^2} = \lim_{A \to \infty} \left( \arctan A - \arctan 1 \right) = \frac{\pi}{4}$.

b) Answer. $\int_1^{\infty} \frac{dx}{1 + x} = \lim_{A \to \infty} \int_1^A \frac{dx}{1 + x} = \lim_{A \to \infty} \left( \ln(1 + A) - \ln 2 \right) = \infty$.

7. Find the sum of the series. If the series does not converge, just write “DIV”. Carefully note the limits of summation.

a) $\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$

Answer. This is nearly the geometric series. Thus, we write

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n = \frac{1}{2} \frac{1}{1 - 2/3} = \frac{3}{2} .$$
b) \[ \sum_{n=3}^{\infty} \frac{1}{n(n-1)} \]

**Answer.** This is a telescoping series:

\[ \sum_{n=3}^{\infty} \frac{1}{n(n-1)} = \sum_{n=3}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = \frac{1}{2}. \]

c) \[ \sum_{n=0}^{\infty} \frac{n}{3^{n-1}} \]

**Answer.** This series suggests the geometric series:

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \]

Now, the factor \( n \) suggests differentiating:

\[ \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}. \]

Now, substitute \( x = 1/3 \):

\[ \sum_{n=0}^{\infty} \frac{n}{3^{n-1}} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4}. \]

d) \[ \sum_{n=0}^{\infty} \frac{(12)^n}{n!} \]

**Answer.** This is just the series for \( e^x \) at \( x = 12 \), so the sum is \( e^{12} \).

8. Consider the hyperbola given by the equation \( x^2 - 2y^2 - 2x + 12y = 138 \).

a) What is its center?  

b) What is the distance between the its vertices?

**Answer.** Complete the square: \( x^2 - 2x + 1 - 2(y^2 - 6y + 9) = 138 + 1 - 18 \), which becomes

\[ (x - 1)^2 - 2(y - 3)^2 = 121 \quad \text{or} \quad \frac{(x - 1)^2}{121} - \frac{(y - 3)^2}{121/2} = 1. \]

Thus the center of the hyperbola is at (1,3), and its major axis is the line \( y = 3 \). Setting \( y = 3 \), we find \( x - 1 = \pm 11 \), so the vertices are at (-10,3),(12,3). Thus the distance between the vertices is 22.
9. Find the area of the region enclosed by the curve given in polar coordinates by \( r = 2 \cos \theta \).

**Answer.** This is the circle of radius 1 centered at the point (1,0) so has area \( \pi \). If you did not recognize the curve, you integrated, but only from 0 to \( \pi \), since that is all you need to traverse the whole boundary. Thus

\[
\text{Area} = \int_{0}^{\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{0}^{\pi} (2 \cos \theta)^2 d\theta = \int_{0}^{\pi} (1 + \cos(2\theta))d\theta = \pi .
\]

10. a) Find the general solution of the differential equation \( y'' - 6y' + 5y = 0 \).

**Answer.** The auxiliary equation, \( r^2 - 6r + 5 = 0 \) has the roots \( r = 1, 5 \). Thus the general solution is

\[
y_h = Ae^x + Be^{5x} .
\]

b) Solve the initial value problem:

\[
y'' - 6y' + 5y = 10 , \quad y(0) = 0, y'(0) = 0 .
\]

**Answer.** A particular solution is the constant function \( y_p = 2 \). Thus the general solution is \( y = y_p + y_h = 2 + Ae^x + Be^{5x} \). We solve for \( A \) and \( B \) from the initial conditions:

\[
0 = 2 + A + B , \quad 0 = A + 5B , \quad \text{so} \quad A = -\frac{5}{2}, \quad B = \frac{1}{2}
\]

and the solution is

\[
y = 2 - \frac{5}{2}e^x + \frac{1}{2}e^{5x} .
\]