1. Find the integrals:

a) \( \int (e^{\sin x})^2 \cos x \, dx \)

**Answer.** Let \( u = \sin x \), \( du = \cos x \, dx \). Then

\[
\int (e^{\sin x})^2 \cos x \, dx = \int (e^u)^2 \, du = \int e^{2u} \, du = \frac{e^{2u}}{2} + C = \frac{e^{2\sin x}}{2} + C.
\]

Alternatively, let \( v = e^{\sin x} \), \( dv = e^{\sin x} \cos x \, dx \), so that

\[
\int (e^{\sin x})^2 \cos x \, dx = \int e^{\sin x} (e^{\sin x} \cos x) \, dx = \int vdv = \frac{v^2}{2} + C = \frac{(e^{\sin x})^2}{2} + C.
\]

b) \( \int x\sqrt{x-1} \, dx \)

**Answer.** Let \( u = x - 1 \) so that \( x = u + 1 \) and \( du = dx \). Then

\[
\int x\sqrt{x-1} \, dx = \int (u+1)^{3/2} \, du = \int (u^{3/2} + u^{1/2}) \, du = \frac{2}{5} (x-1)^{5/2} + \frac{2}{3} (x-1)^{3/2} + C.
\]

2. Integrate \( \int \frac{t^2}{(t^2-1)(t-2)} \, dt \)

**Answer.** We expand the function in partial fractions. The roots are -1, 1, 2, so we write

\[
\frac{t^2}{(t^2-1)(t-2)} = \frac{A}{t+1} + \frac{B}{t-1} + \frac{C}{t-2} = \frac{A(t-1)(t-2) + B(t+1)(t-2) + C(t^2-1)}{(t^2-1)(t-2)}.
\]

Equate the numerators at the roots.

- \( t = -1 \): \((-1)^2 = A(-2)(-3)\) so \( A = \frac{1}{6} \)
- \( t = 1 \): \(1^2 = B(2)(-1)\) so \( B = -\frac{1}{2} \)
- \( t = 2 \): \(2^2 = C(4-1)\) so \( C = \frac{4}{3} \)

This gives us

\[
\int \frac{t^2}{(t^2-1)(t-2)} \, dt = \frac{1}{6} \int \frac{dt}{t+1} - \frac{1}{2} \int \frac{dt}{t-1} + \frac{4}{3} \int \frac{dt}{t-2} = \frac{1}{6} \ln(t+1) - \frac{1}{2} \ln(t-1) + \frac{4}{3} \ln(t-2) + C.
\]

3. Integrate \( \int x\ln x \, dx \)

**Answer.** We integrate by parts so as to get rid of the \( \ln \) term: \( u = \ln x \), \( dv = x \, dx \), so that \( du = dx/x \), \( v = x^2/2 \), and

\[
\int x\ln x \, dx = \frac{x^2}{2} \ln x - \int \left( \frac{x^2}{2} \right) \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.
\]
4. A certain compound transforms from state \( A \) to state \( B \) at a (per minute) rate proportional to the concentration of \( B \) in the mixture:

\[
\frac{dc_A}{dt} = -0.02c_B,
\]

where \( c_A \) and \( c_B \) are the concentrations of \( A \) and \( B \) respectively (and, assuming no other material is present, \( c_A + c_B = 1 \)). If at time \( t = 0 \) the mixture is 90% in state \( A \) how long will it take to be 10% \( A \)?

**Answer.** Substitute \( c_B = 1 - c_A \) in the differential equation, and separate variables, obtaining

\[
\frac{dc_A}{1-c_A} = -0.02dt.
\]

Integrate both sides and exponentiate:

\[-\ln(1-c_A) = -0.02t + C \quad \text{exponentiating to} \quad 1-c_A = Ke^{0.02t}.\]

Solve for \( K \) using \( c_A = .9 \) when \( t = 0 \), getting \( K = .1 \). Now solve for \( c_A \):

\[c_A = 1 - .1e^{0.02t}.\]

(Of course this makes sense so long as \( c_A > 0 \); once \( A \) is gone, the process stops.) Now set \( c_A = .1 \) and solve for \( t \): \( .1e^{0.02t} = .9 \), so

\[t = \frac{\ln 9}{0.02} = 109.86 \text{ minutes}.\]

5. Find the limit. Show your work.

a) \( \lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} = \)

**Answer.** At \( x = 1 \), both numerator and denominator are zero, so l’Hôpital’s rule applies:

\[\lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} = \lim_{x \to 1} \frac{1/x}{\pi \cos(\pi x)} = -\frac{1}{\pi}.\]

b) \( \lim_{x \to 0} \frac{xe^x}{e^{2x} - 1} = \)

**Answer.** Again both numerator and denominator are zero at \( x = 0 \), so

\[\lim_{x \to 0} \frac{xe^x}{e^{2x} - 1} = \lim_{x \to 0} \frac{e^x + xe^x}{2e^{2x}} = \frac{1}{2}.\]

c) \( \lim_{x \to \infty} \frac{3x^6 + 7x^4}{2(x^3 + 1)^2} = \frac{3}{2} \)

since the factors have the same degree.

6. Do the integrals converge? If so, evaluate:

a) **Answer.**

\[
\int_0^\infty xe^{-x}dx = \lim_{A \to \infty} \int_0^A xe^{-x}dx = \lim_{A \to \infty} (xe^{-x} - e^{-x})|_0^A = \lim_{A \to \infty} (e^{-A}(A-1)-(1)) = 1.
\]
b) \( \int_2^\infty \frac{dx}{x(\ln x)^{23}} \)

**Answer.** Let \( u = \ln x, \ du = dx/x. \) Then

\[
\int_2^A \frac{dt}{x(\ln x)^{23}} = \int_\ln 2^{A}\frac{du}{u^{23}} = \frac{-u^{-24}}{24} \mid_{\ln 2}^{A} = \frac{1}{24A} \left( \frac{1}{(\ln 2)^{24}} - \frac{1}{(\ln A)^{24}} \right)
\]

which converges to \( 1/(24(\ln 2)^{24}) \) as \( A \to \infty. \)

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7. Do the series converge or diverge? Give a valid reason for your answer.

a) \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{(n+1)^3} \)

**Answer.** The series diverges by comparison with the \( p \)-test with \( p = 1: \) the denominator is only of degree 1 more than the numerator.

b) \( \sum_{n=1}^{\infty} \frac{\ln n}{2^n} \)

**Answer.** The series converges by comparison with a geometric series:

\[
\frac{\ln n}{2^n} \leq \frac{2^{n/2}}{2^n} = \left( \frac{1}{\sqrt{2}} \right)^n,
\]

and \( 1/\sqrt{2} < 1. \)

c) \( \sum_{n=1}^{\infty} \frac{(n!+1)^2}{((n+1)!)^2} \)

**Answer.** The series converges by comparison with the \( p \)-test with \( p = 2. \) Divide both numerator and denominator by \( (n!)^2: \)

\[
\frac{(n!+1)^2}{((n+1)!)^2} = \frac{(1 + \frac{1}{n})^2}{(\frac{(n+1)!}{n!})^2} \leq \frac{2}{(n+1)^2}
\]

since the numerator is bounded by 2.

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8. Find the vertices of the conic given by the equation \( 4x^2 - y^2 + 8x - 4y + 12 = 0 \)

**Answer.** Complete the square:

\[
4(x^2 + 2x + 1) - (y^2 + 4y + 4) + 12 - 4 + 4 = 0 \quad \text{or} \quad 4(x+1)^2 - (y+2)^2 = -12.
\]

This gives the standard form

\[
\frac{-(x+1)^2}{3} + \frac{(y+2)^2}{12} = 1.
\]

This is a hyperbola with center at \((-1,-2)\) and axis the line \( x = -1. \) Setting \( x = -1 \) gives the \( y \) coordinates of the vertices:

\[
\frac{(y+2)^2}{12} = 1 \quad \text{or} \quad y = -2 \pm \sqrt{12}.
\]

Thus the vertices are at \((-1, -2 - 2\sqrt{3})\) and \((-1, -2 + 2\sqrt{3}).\)

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9. Find the area of the region enclosed by the curve given in polar coordinates by \( r = 2e^{\theta}, \ 0 \leq \theta \leq 2\pi \) and the segment of the \( x \) axis between \( x = 2 \) and \( x = 2e^{2\pi}. \)
**Answer.** From the diagram we see that the area is

\[
\text{Area} = \frac{1}{2} \int_{0}^{2\pi} r^2 d\theta = 2 \int_{0}^{2\pi} e^{2\theta} d\theta = e^{2\theta} \bigg|_{0}^{2\pi} = e^{4\pi} - 1.
\]

10. a) Find the general solution of the homogeneous differential equation \( y'' - 3y' + 2y = 0 \).

**Answer.** The roots of the equation \( r^2 - 3r + 2 = 0 \) are 1, 2. Thus the general solution is

\[ y_h = Ae^x + Be^{2x}. \]

b) Find a particular solution of the homogeneous differential equation \( y'' - 3y' + 2y = \sin x \).

**Answer.** We use the method of undetermined coefficients. Try a solution of the form \( y = A \cos x + B \sin x \):

\[
(-A \cos x - B \sin x) - 3(-A \sin x + B \cos x) + 2(A \cos x + B \sin x) = \sin x,
\]

leading to the equations \(-A - 3B + 2A = 0\), \(-B - 3A + 2B = 1\). The solutions are \( B = 1/10 \), \( A = 3/10 \), so a particular solution is

\[ y_p = 0.3 \cos x + 0.1 \sin x. \]