Find all the integrals. Remember that definite integrals should have numerical answers. You MUST show your work.

1a. \( \int xe^x \, dx \)

**Answer.** Integrate by parts with \( u = x \), \( du = dx \), \( dv = e^x \, dx \), \( v = e^x \):

\[
\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.
\]

1b. \( \int xe^x^2 \, dx \)

**Answer.** Make the substitution \( u = x^2 \), \( du = 2x \, dx \):

\[
\int xe^x^2 \, dx = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C.
\]

2. \( \int_2^4 \frac{xdx}{x^2 - 4} \)

**Answer.** First, substitution works: for \( u = x^2 - 4 \), \( du = 2x \, dx \), we get

\[
\int \frac{xdx}{x^2 - 4} = \frac{1}{2} \int \frac{du}{u}.
\]

At \( x = 2, u = 12 \) and at \( x = 2, u = 0 \), so we have

\[
\int_2^4 \frac{xdx}{x^2 - 4} = \frac{1}{2} \int_0^{12} \frac{du}{u} = \frac{1}{2} \ln u \bigg|_0^{12} = \ln(12) - \ln(0).
\]

If you got this far you will get full credit. Since \( \ln(0) \) is undefined the limit does not exist. Actually, there was a typo in the exam; the lower limit should have been 3, in which case, the answer would be \( \sqrt{12} \).

This can also be done by the partial fractions expansion. Since \( x^2 - 4 = (x - 2)(x + 2) \), we can write

\[
\frac{x}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2}
\]

for some \( A \) and \( B \). Putting the expression on the right over a common denominator, we must have equality of the numerators: \( x = A(x + 2) + B(x - 2) \). We find \( A \) and \( B \) by evaluating at the roots 2, \( -2 \): \( A = 1/2, \, B = 1/2 \). Thus

\[
\int_2^4 \frac{xdx}{x^2 - 4} = \frac{1}{2} \int_2^4 \left( \frac{1}{x - 2} + \frac{1}{x + 2} \right) dx = \frac{1}{2} \left[ \ln(x - 2) + \ln(x + 2) \right]_2^4.
\]

At this point, we have the same problem as above; since 2-2=0, we can’t evaluate \( \ln 0 \).

3. \( \int_0^{\pi/4} \tan x \ln(\cos x) \, dx \)

**Answer.** The substitution \( w = \ln(\cos x), \, dw = -\tan x dx \) leads to:

\[
\int_0^{\pi/4} \tan x \ln(\cos x) \, dx = -\int_0^{\ln(1/\sqrt{2})} \frac{wdw}{w}.
\]
We could also try integration by parts: if we let \( dv = \ln(\cos x) \, dx \), we can't integrate, so we let \( du = \tan x \, dx \), giving \( v = -\ln(\cos x) \). Then \( u = \ln(\cos x) \), \( du = -\tan x \, dx \). We then get

\[
\int \tan x \ln(\cos x) \, dx = -\frac{1}{2}(\ln(\sqrt{2}/2))^2 - \int \tan x \ln(\cos x) \, dx
\]

so

\[
\int \tan x \ln(\cos x) \, dx = -\frac{1}{2}(\ln(\sqrt{2}/2))^2 .
\]

Now, evaluating at the limits, we get

\[
\int_{0}^{\pi/4} \tan x \ln(\cos x) \, dx = -\frac{1}{2}(\ln(\sqrt{2}/2))^2 = -\frac{1}{8}(\ln 2)^2 .
\]

You could also remember that \( \tan x = \sin x / \cos x \) suggesting the substitution

\[
u = \cos x, \quad dv = -\sin x \, dx .
\]

When \( x = 0 \), \( u = 1 \) and when \( x = \pi/4 \), \( u = 1/\sqrt{2} \). Thus

\[
\int_{0}^{\pi/4} \tan x \ln(\cos x) \, dx = -\int_{0}^{1/\sqrt{2}} \ln u \, du .
\]

Now make the substitution \( w = \ln u \), \( dw = du/u \), getting

\[
-w \int_{0}^{\ln(1/\sqrt{2})} w \, dw = -\frac{w^2}{2} \bigg|_{0}^{\ln(1/\sqrt{2})} .
\]

Now, \( \ln(1/\sqrt{2}) = -\ln 2/2 \), so the answer is

\[
\int_{0}^{\pi/4} \tan x \ln(\cos x) \, dx = -\frac{1}{2}(\ln 2)^2 = -\frac{1}{8}(\ln 2)^2 .
\]

4. \( \int \frac{x}{\sqrt{1-x^2}} \, dx \)

Answer. Let \( u = 1-x^2 \), \( du = -2x \, dx \), so

\[
\int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int u^{-1/2} \, du = -\frac{1}{2}(2u^{1/2}) + C = -\sqrt{1-x^2} + C .
\]

We could also make the substitution \( x = \sin u \), \( dx = \cos u \, du \), \( \sqrt{1-x^2} = \cos u \). We get

\[
\int \frac{x}{\sqrt{1-x^2}} \, dx = \int \sin u \, du = -\cos u + C = -\sqrt{1-x^2} + C .
\]

5. \( \int_{2}^{4} \frac{dx}{x(x^2 - 1)} \)

Answer. We consider the partial fractions expansion. Since \( x(x^2 - 1) = x(x-1)(x+1) \), we have

\[
\frac{1}{x(x^2 - 1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2 - 1) + B(x+1) + Cx(x-1)}{x^2 + 1} .
\]

Setting \( x = 0 \) and equating numerators, we get \( A = -1 \). For \( x = 1 \), we get \( B = 1/2 \) and for \( x = -1 \), we get \( C = 1/2 \). We can now integrate;

\[
\int_{2}^{4} \frac{dx}{x(x^2 - 1)} = -\int_{2}^{4} \frac{dx}{x} + \frac{1}{2} \int_{2}^{4} \frac{dx}{x-1} + \frac{1}{2} \int_{2}^{4} \frac{dx}{x+1} = -\ln x + \frac{1}{2} \ln(x^2 - 1) \bigg|_{2}^{4}
\]

\[
= -\ln 4 + \frac{1}{2} \ln(15) + \ln 2 - \frac{1}{2} \ln 3 = \ln(\frac{\sqrt{5}}{2}) = 1.118 .
\]