Calculus I
Practice Problems 3: Answers

1. A point moves around the unit circle so that the angle it makes with the x-axis at time \( t \) is \( \theta(t) = (t^2 + t)\pi \). Let \((x(t), y(t))\) be the cartesian coordinates of the point at time \( t \). What is \( dy/dt \) when \( t = 3 \)?

**Answer.** \( y(t) = \sin((t^2 + t)\pi) \), so 
\[
\frac{dy}{dt} = \cos((t^2 + t)\pi)(2t + 1)\pi.
\]
Evaluating at \( t = 3 \): 
\[
\frac{dy}{dt} = \cos(10\pi)(2(3) + 1) = (2(3) + 1)\pi = 7\pi.
\]

2. Find the derivative: \( f(x) = \sin x \cos x \)

**Answer.** 
\[
f'(x) = \sin x(-\sin x) + \cos x\cos x = \cos^2 x - \sin^2 x.
\]

3. Find the derivative: \( g(x) = \frac{\sin x}{\cos x} \)

**Answer.** This is \( f(x) = \tan x \), so its derivative is \( f'(x) = \sec^2 x \). If you use the quotient rule, you get 
\[
f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}.
\]

4. Let \( f(x) = x \sin x \). Find the equation of the tangent line to the graph \( y = f(x) \) at the points \( x = (n + 1/2)\pi \) for any integer \( n \).

**Answer.** The easy answer is to draw the graph and observe that the tangent line is \( y = x \). See the graph. However, since the slope of the tangent line is given by the derivative, we calculate: \( f'(x) = x \cos x + \sin x \), and evaluate at \( x = (n + 1/2)\pi \), finding \( f'((n + 1/2)\pi) = 1 \). When \( x = (n + 1/2)\pi \), we calculate that \( y = (n + 1/2)\pi \) also, so the tangent line has the equation 
\[
\frac{y - (n + 1/2)\pi}{x - (n + 1/2)\pi} = 1, \quad \text{or} \quad y = (-1)^n x.
\]
5. Consider the curves \( C_1 : y = \sin x \) and \( C_2 : y = \cos x \).

a) At which points \( x \) between \(-\pi/2\) and \( \pi/2\) do the curves have parallel tangent lines?

b) At which such points do they have perpendicular tangent lines?

Answer. At the point \( x \), the tangents to the curves \( C_1 \) and \( C_2 \) have slope \( \cos x, -\sin x \) respectively.

a) These lines are parallel if \( \cos x = -\sin x \), or \( \tan x = -1 \), which has the solution \( x = -\pi/4 \).

b) These lines are perpendicular if \( \cos x (-\sin x) = -1 \), or \( \sin x \cos x = 1 \). But then

\[
\sin(2x) = 2\sin x \cos x = 2
\]

which has no solution: the curves never perpendicular tangent lines. Here are the graphs of the given curves.
6. Differentiate: \( f(x) = \frac{1 + \tan x}{1 - \tan x} \)

**Answer.** Use the addition formula for the tangent: \( f(x) = \tan(x + \pi/4) \). Then differentiate: \( f'(x) = \sec^2(x + \pi/4) \). If you used the quotient rule, you probably ended up with

\[
f'(x) = \frac{2\sec^2 x}{(1 - \tan x)^2},
\]

which is also the correct answer.

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7. Let \( y = x + 25x^{-1} \). Find an approximate value of \( y \) when \( x = 3.2 \).

**Answer.** If we start at \( x = 3 \), we find \( y = 3 + 25/3 = 11.33 \). Take the increment \( dx = 0.2 \), and now take differentials. Take the increment \( dx = 0.2 \) and now take differentials:

\[
dy = dx - 25x^{-2} dx.
\]

Substituting the values determined above: \( dy = .2 - (25/9)(.2) = -.36 \), so the approximate value of \( y \) is 11.33.36 = 10.98. Note that at \( x = 5 \) we have \( dy = 0 \), so this technique will not work to approximate values of \( y \) for \( x \) near 5.

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8. Find an approximate value of \( \tan(0.26\pi) \).

**Answer.** Here we want to start at \( x = \pi/4, y = 1 \) and \( dx = .01\pi \). We have \( dy = \sec^2 x dx \), so at \( x = \pi/4 \), \( dy = (\sqrt{2})^2(.01) = .02 \). Thus the approximation to \( y \) is 1+.02=1.02.

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9. Find the equation of the tangent line to \( y = x^2(x^3 - 1) \) at \( (2,28) \).

**Answer.** Taking differentials,

\[
dy = 2x(x^3 - 1)dx + x^2(3x^2 dx).
\]

Since this gives the linear approximation to the graph, we get the equation of the tangent line by substituting \( x = 2, dx = x = 2, dy = y = 28 \):

\[
y = 28 = (4)(7)(x - 2) + 4(12)(x - 2)
\]

which simplifies to \( y = 76x - 124 \).

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10. Find the equation of the tangent line to the curve \( y = x \cos x \) at \( (\pi/4, \pi \sqrt{2}/8) \).

**Answer.** The equation of differentials is \( dy = -x \sin x dx + \cos x dx \). Substituting \( x = \pi/4, dx = x - \pi/4 \), \( dy = y - \pi \sqrt{2}/8 \):

\[
y - \frac{\pi \sqrt{2}}{8} = \frac{\pi \sqrt{2}}{4} \left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right)
\]

which simplifies to

\[
y = \frac{\sqrt{2}}{2} \left(1 - \frac{\pi}{4}\right) x + \frac{\pi^2 \sqrt{2}}{32}
\]