Calculus I

Practice Problems 10: Answers

1. \( \int_1^3 (2t + 1)^3 \, dt = \)

**Answer.**

\[
\int_1^3 (2t + 1)^3 \, dt = \frac{1}{2} \int_3^7 u^3 \, du = \frac{1}{2} \left[ \frac{u^4}{4} \right]_3^7 = \frac{1}{8}(7^4 - 3^4)
\]

2. \( \int_{-1}^1 (4x^3 - 2x^2 + 1) \, dx = \)

**Answer.** Since \( x^3 \) is an odd function and the domain is symmetric about 0, the first term contributes nothing. Thus the integral is equal to

\[
\int_{-1}^1 (-2x^2 + 1) \, dx = (-2x^3/3 + x) \bigg|_{-1}^{1} = \frac{2}{3}
\]

3. Calculate the definite integrals:

a) \( \int_{-4}^4 (x^2 - 3 + \cos x) \, dx \)

**Answer.** Since this is an even function and the domain is symmetric about 0, the integral is

\[
2 \int_0^4 (x^2 - 3 + \cos x) \, dx = 2 \left[ \frac{x^3}{3} - 3x + \sin x \right]_0^4 = 2 \left( \frac{64}{3} - 12 + \sin(4) \right) = \frac{56}{3} + 2\sin(4).
\]

b) \( \int_0^{\pi/4} \frac{\sin x}{\cos^3 x} \, dx \)

**Answer.** Let \( u = \cos x, \, du = -\sin x \, dx \). When \( x = 0, u = 1 \) and when \( x = \pi/4, u = \sqrt{2}/2 \). Thus

\[
\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} \, dx = -\frac{1}{2} \int_1^{\sqrt{2}/2} u^{-3} \, du = \frac{1}{2} \left. u^{-2} \right|_{\sqrt{2}/2}^{1} = \frac{1}{2} \left( \frac{1}{2/4} - 1 \right) = \frac{1}{2}.
\]

4. Integrate:

a) \( \int_1^{4} \frac{1}{\sqrt{y}((\sqrt{y} + 1)^2} \, dy \)

**Answer.** Let \( u = y^{1/2}, \, du = (1/2)y^{-1/2} \, dy \). When \( y = 1, \, u = 1 \) and when \( y = 4, \, u = 2 \). Thus

\[
\int_1^{4} \frac{1}{\sqrt{y}((\sqrt{y} + 1)^2} \, dy = 2 \int_1^{2} \frac{du}{(u+1)^2} = -2(u+1)^{-2} \bigg|_{1}^{2} = -2 \left[ \frac{1}{3} - \frac{1}{2} \right] = \frac{1}{3}.
\]

b) \( \int_0^{\pi/2} \cos^2 x \sin x \, dx = \)

**Answer.**

\[
\int_0^{\pi/2} \cos^3 x \, dx = \frac{1}{3}
\]
5. Evaluate

a) \( \frac{d}{dx} \int_0^{2x+1} \cos t \, dt \)

**Answer.** Let \( u = 3x + 1 \). By the fundamental theorem of the calculus \( \frac{d}{du} \int_0^u \cos t \, dt = \cos u \). Now, by the chain rule

\[
\frac{d}{dx} \int_0^{2x+1} \cos t \, dt = \left( \frac{d}{du} \int_0^u \cos t \, dt \right) \frac{du}{dx} = (\cos u)(2) = 2\cos(2x+1) .
\]

b) \( \frac{d}{dx} \int_0^x t^3 \, dt \)

**Answer.** Let \( u = x^2 \). By the fundamental theorem of the calculus \( \frac{d}{du} \int_0^u t^3 \, dt = u^3 \). Now, by the chain rule

\[
\frac{d}{dx} \int_0^x t^3 \, dt = \left( \frac{d}{du} \int_0^u t^3 \, dt \right) \frac{du}{dx} = u^3(2x) = (x^2)^3(2x) = 2x^7 .
\]

6. Find the area of the region in the right half plane \((x > 0)\) bounded by the curves \( y = x - x^3 \) and \( y = x^2 - x \).

**Answer.** First, we find the points of intersection of the curves by solving the equation \( x - x^3 = x^2 - x \). This becomes \( x^3 + x^2 - 2x = 0 \), which has the solutions \( x = -2, 0, 1 \). Since we are interested only in \( x > 0 \), the range of integration is the interval \((0, 1)\). From the graph (see the figure), the cubic curve lies above the quadratic, so the area is

\[
\int_0^1 [(x - x^3) - (x^2 - x)] \, dx = \int_0^1 (-x^3 - x^2 + 2x) \, dx = -\frac{1}{4} - \frac{1}{3} + 1 = \frac{5}{12} .
\]

7. Find the area of the region in the first quadrant bounded by the curves \( y = \sin \frac{\pi}{2} x \) and \( y = x \).

**Answer.** The curves intersect at \( x = 0, 1 \), and the sine curve is above the line (see the figure), so the area is

\[
\int_0^1 (\sin \frac{\pi}{2} x - x) \, dx = \frac{2}{\pi} \left( -\cos \frac{\pi}{2} x \right) \Big|_0^1 = \left( \frac{2}{x} \right) \Big( \frac{\pi}{2} - 1 \right) - \left( \frac{2}{x} \right) \Big( -1 - 0 \right) = \frac{2}{x} - \frac{1}{2} .
\]
8. Find the area of the region under the curve $y = x\sqrt{x^2 + 1}$, above the $x$-axis and bounded by the lines $x = 1$ and $x = 3$.

**Answer.** The area (see the figure) is given by \( \int_1^3 x\sqrt{x^2 + 1} \, dx \). Let \( u = x^2 + 1 \), \( du = 2x \, dx \). When \( x = 1 \), \( u = 2 \) and when \( x = 3 \), \( u = 10 \). This substitution leads to:

\[
\int_1^3 x\sqrt{x^2 + 1} \, dx = \frac{1}{2} \int_2^{10} u^{1/2} \, du = \frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_2^{10} = \frac{1}{3} (10\sqrt{10} - 2\sqrt{2}).
\]

9. Find the area under the curve $y = x^2 + x^{-2}$, above the $x$-axis and between the lines $x = 1$ and $x = 2$.

**Answer.** The area is

\[
\int_1^2 (x^2 + x^{-2}) \, dx = \left( \frac{x^3}{3} - x^{-1} \right)_1^2 = \left( \frac{8}{3} - \frac{1}{2} \right) - \left( \frac{1}{3} - 1 \right) = \frac{17}{6}.
\]
10. What is the area of the region bounded by the curves \( y = x^3 - x \) and \( y = 3x \)?

**Answer.** First find the points of intersection:

\[ x^3 - x = 3x \quad \text{or} \quad x^3 = 4x \]

has the solutions \( x = -2, 0, 2 \). The line \( y = 3x \) lies below the curve \( y = x^3 - x \) in the interval \((-2, 0)\) and above that curve in the interval \((0,2)\) (see the accompanying figure). The areas of these two regions are given by the integrals:

\[
\int_{-2}^{0} [(x^3 - x) - 3x] \, dx , \quad \int_{0}^{2} [3x - (x^3 - x)] \, dx .
\]

Since the two intervals are symmetric about 0, and the integrand is an odd function, these two integrals are the same. Thus the area is

\[
2 \int_{0}^{2} [3x - (x^3 - x)] \, dx = 2 \int_{0}^{2} (4x - x^3) \, dx = 2(2x^2 - \frac{x^4}{4}) \bigg|_{0}^{2} = 2(8 - 16/4) = 8 .
\]