1. A curve $C$ in the plane is the graph of the relation $x^3 - xy^2 + y^2 + x^2 = 2$. Find the equation of the tangent line to the curve at the point $(-1, 2)$.

**Answer.** Take the differential of the defining equation:

$$3x^2 dx - y^2 dx - 2xy dy + dy + 2dx = 0.$$ 

Substitute $x = -1$, $y = 2$:

$$3dx - 4dx + 4dy + dy + 2dx = 0$$

which simplifies to $dx + 5dy = 0$.

Since $dx$ and $dy$ represent increments on the tangent line, replace them by $x - (-1)$ and $y - 2$ to obtain the equation of the tangent line:

$$(x + 1) + 5(y - 2) = 0$$

which simplifies to $x + 5y = 9$ or $y = (-x + 9)/5$.

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2. A lamp is being lowered down a vertical pole at a rate of 3 ft/sec. A 6 foot man stands 20 feet away from the pole. At what rate is the shadow of the man lengthening when the lamp is 56 feet off the ground?

**Answer.** Draw the diagram (see below); letting $x$ be the length of the shadow, and $y$ the distance of the lamp from the ground. We first find the relation of $x$ and $y$ at any time. By similar triangles,

$$\frac{6}{y} = \frac{x}{20+x}$$

which simplifies to $120 + 6x = xy$.

Now, as the lamp falls, $x$ and $y$ are functions of time $t$. Since we want to relate the rates of change, we differentiate this expression with respect to $t$:

$$\frac{dx}{dt} = \frac{y \frac{dy}{dt} + x \frac{dx}{dt}}{6}.$$  

Now, when the lamp is 56 feet off the ground, the length of the shadow is found by solving

$$120 + 6x = x(56)$$

giving $x = 12/5$.

Substituting these values (and $\frac{dy}{dt} = -3$) into (1):

$$\frac{dx}{dt} = \frac{12}{5}(-3) + 56 \frac{dx}{dt},$$

which gives $\frac{dx}{dt} = 36/250 = .144$ ft/sec.
3. Let \( y = (x^2 - 1)(x^2 - 5) \). For what value of \( x \) in the interval \([-2, 2]\) is \( y \) a maximum? a minimum? Find the points of inflection of the graph.

**Answer.** Let \( f(x) \) represent the function defining \( y \). Differentiating:

\[
f'(x) = 2x(x^2 - 5) + (x^2 - 1)(2x) = 2x^3 - 12x + 2x^3 - 2x = 4x^3 - 12x,
\]

\[
f''(x) = 12x^2 - 12.
\]

The critical points are \( 0, \pm \sqrt{3} \). We must now evaluate the function at the endpoints and critical points to find the maximum and minimum. Since the function is even, we need only check positive values of \( x \). Since \( f'(x) < 0 \) in \((0, \sqrt{3})\) and \( f'(x) > 0 \) in \((\sqrt{3}, 2]\), the minimum has to be at \( x = \sqrt{3} \) (and thus also at \( x = -\sqrt{3} \)). To find the maximum, we compare values at the endpoints and at 0. Since \( y = -3 \) at the endpoints, and \( y = 5 \) at \( x = 0 \), the maximum is at \( x = 0 \). The points of inflection are at the solutions of \( f''(x) = 0 \), so are at \( x = \pm 1 \). See the figure for a graph of the function.

![Graph of the function](image)

4. I have to make a closed cylindrical can to hold 12 cu. ft. The material to make the top and bottom costs $6 a sq. ft., and the material to make the side costs $10 per sq. ft. What are the dimensions which minimize the cost? (It will suffice to give either the radius of the base or the height).

**Answer.** Let \( r \) be the radius of the base, and \( h \) the height of a typical cylinder. We know that the volume must be \( V = \pi r^2 h = 12 \). Now the cost is $10 for the top and bottom; that contributes $10(2)\pi r^2$, and $6$ for the sides, contributing $6(2\pi rh)$. Thus, the total cost is

\[
C = 20\pi r^2 + 6(2\pi rh) = 20\pi r^2 + 144r^{-1},
\]

using the equation \( \pi r^2 h = 12 \) to express \( h \) in terms of \( r \). Differentiating,

\[
C = 40\pi r - 144r^{-2}.
\]

Since we are looking for a minimum, we set this equal to 0. This gives

\[
r^3 = \frac{144}{40\pi}
\]
so \( r = 1.046 \) and \( h = 3.488 \).

5. Graph

\[
y = \frac{x(x - 2)}{x^2 - 1} = \frac{x^2 - 2x}{x^2 - 1}
\]

You must show enough work to explain how you found the various features of the graph.

**Answer.** Since the numerator and denominator have the same degree, the horizontal asymptote is \( y = 1 \). The vertical asymptotes are at \( x = \pm 1 \). Now, differentiating:

\[
y' = \frac{(x^2 - 1)(2x - 2) - (x^2 - 2x)(2x)}{(x^2 - 1)^2} = \frac{2}{(x^2 - 1)^2}(x^2 - x + 1).
\]

This is always positive, so the graph is always increasing. This information suffices to give the graph. See the figure.