1. A curve in the plane is given implicitly by the equation

\[ 2x^2 + 2xy + 3y^2 = 40 \, . \]

At what points does the curve have a horizontal tangent line?

**Solution.** We differentiate implicitly by taking the differential of both sides:

\[ 4xdx + 2xdy + 2ydx + 6ydy = 0 \, \text{which simplifies to} \, (4x + 2y)dx + (2x + 6y)dy = 0 \, . \]

We see that we must have \( dy/dx = 0 \) (the condition for a horizontal tangent) when \( 4x + 2y = 0 \), or \( y = -2x \). Substituting that in the equation of the curve, we get

\[ 2x^2 - 4x^2 + 12x^2 = 40 \, \text{or} \, 10x^2 = 40 \, , \]

so that \( x = \pm 2 \) and thus \( y = \mp 4 \). The points at which there is a horizontal tangent are thus (2,-4) and (-2,4).

If the issue were to find the points at which the tangent line is vertical, then we set the coefficient of \( dy \) equal to zero, obtaining \( x = -3y \), with the solutions \( \pm(-3\sqrt{8/3}, \sqrt{8/3}) \).

2. A pool filled with water is shaped like a box lying over a rectangle of area 60 ft\(^2\). Because of a break in the bottom, water begins to leak out of the pool at a rate proportional to the volume \( V \) of water in the pool according to the formula

\[ \frac{dV}{dt} = \frac{1}{20} V \, . \]

\( V \) is measured in ft\(^3\) and time in minutes. At what rate is the height of the water in the pool decreasing when the height is 6 feet? Remember that the volume of water is equal to the area of the base times the height of the water.

**Solution.** The relationship between volume and height is \( V = 60h \), since the area of the base is 60 ft\(^2\). Thus \( 60dh/dt = dV/dt \). Then,

\[ \frac{dh}{dt} = \frac{1}{60} \frac{dV}{dt} = \frac{1}{60} \frac{1}{20} V = \frac{1}{60} \frac{1}{20} 60h = \frac{1}{20} h = \frac{6}{20} = .3 \text{ feet/min when } h \text{ is 6 feet.} \]

3. What point on the line \( 2x + y = 10 \) is closest to the origin?
Solution. The distance of \((x, y)\) from the origin is \(\sqrt{x^2 + y^2}\). Thus we want to minimize \(x^2 + y^2\) subject to the condition \(y = 10 - 2x\). We take \(f(x) = x^2 + (10 - 2x)^2\); the minimum of \(f\) is to be found among the points at which \(f'(x) = 0\). Now

\[
f'(x) = 2x + 2(10 - 2x)(-2) = 10x - 40 = 0
\]

which has the solution \(x = 4\). At \(x = 4\), \(y = 10 - 2(4) = 10 - 8 = 2\), so the answer is \((4, 2)\).

4. What is the maximum of \(y = \frac{x}{x^2 + 1}\)?

Solution. We calculate the derivative:

\[
\frac{dy}{dx} = \frac{x^2 + 1 - x(2x)}{(x^2 + 1)^2}
\]

This is zero when \(x^2 + 1 - x(2x) = 1 - x^2 = 0\), so at the points \(x = \pm 1\). For \(x = 1\), \(y = 1/2\) and for \(x = -1\), \(y = -1/2\), thus the maximum value is 1/2.

To be perfectly precise, we need to verify that the function has a maximum. But, since \(|x| < x^2 + 1\) for all \(x\), and \(y \to 0\) as \(|x| \to \infty\), we must have a point at which the maximum is achieved.

5. Graph \(y = \frac{x^2}{x^2 - 1}\)

showing clearly in what intervals the graph is increasing and decreasing.

Solution. First of all, the graph will have vertical asymptotes at \(x = \pm 1\). To find out where it is increasing or decreasing, we calculate the derivative:

\[
\frac{dy}{dx} = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2}.
\]

Thus \(dy/dx > 0\) for \(x < 0\) and \(dy/dx < 0\) for \(x > 0\). The curve is thus increasing in the left half plane, and decreasing in the right half plane.