Sequences and the Difference Operator

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1 Sequences

Number sequences appear all throughout mathematics. Examples are:

- $1, 2, 3, 4, 5, 6, \ldots$ (the natural numbers)
- $1, 4, 9, 16, 25, 26, \ldots$ (the square numbers)
- $3, 9, 27, 81, 243, 729, \ldots$ (the powers of 3)
- $1, 2, 3, 5, 8, 13, \ldots$ (the Fibonacci numbers)
- $1, 2, 4, 7, 12, 20, \ldots$ (shifted Fibonacci numbers)
- $1, 2, 8, 42, 262, 1828, \ldots$ (the meandric numbers)

Often we give the sequence a name or label so that we can easily reference it later. For instance, we may decide that $a_n$ will represent the $n$th natural number. That is,

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 5, a_6 = 6, \ldots$$

and in general

$$a_n = n.$$

Let $b_n$ represent the $n$th square number. Then

$$b_1 = 1, b_2 = 4, b_3 = 9, b_4 = 16, b_5 = 25, b_6 = 36, \ldots$$

and in general

$$b_n = n^2.$$

Let $c_n$ represent the $n$th power of 3. Then

$$c_1 = 3, c_2 = 9, c_3 = 27, c_4 = 81, c_5 = 243, c_6 = 729, \ldots$$

and in general

$$c_n = 3^n.$$
Notice that, using this formula for \( c_n \), it makes sense to talk about \( c_0 \), that is, the 0th power of 3:

\[
c_0 = 3^0 = 1.
\]

The same is true for \( a_n \) and \( b_n \):

\[
a_0 = 0 \text{ and } b_0 = 0^2 = 0.
\]

These sequences are nice because we can write down a formula for the \( n \)th element in the sequence. That is, given the formula I can immediately calculate the \( n \)th element of the sequence. For instance, if I want to know the 277th square number, I simply compute

\[
b_{277} = 277^2 = 76729
\]

or if I want to know the 13th power of 3, I compute

\[
c_{13} = 3^{13} = 1594323
\]

The Fibonacci numbers are a little different. I begin by giving them a name, say \( f_n \) for the \( n \)th Fibonacci number. Then

\[
f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, \ldots
\]

But how do I write down a formula for this sequence? Right now it is not obvious. However, it is obvious that

\[
f_1 = 1, f_2 = 2, \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n > 2.
\]

This is not a formula in the sense that I can immediately calculate the \( n \)th Fibonacci number from the sequence – the calculation requires knowledge of the previous two Fibonacci numbers. The equation \( f_n = f_{n-1} + f_{n-2} \) is called a recurrence relation for the Fibonacci numbers. In a sense, it is the next best thing to a formula – given the first couple of terms of the sequence, I can compute the rest rather quickly.

One of the things we will explore in these lectures is the following: given a recurrence relation for a sequence, can we derive its formula? The answer is sometimes yes. In the case of Fibonacci numbers, it turns out that

\[
f_n = \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1} \sqrt{5}}.
\]

Call the \( n \)th shifted Fibonacci number \( g_n \). Then \( g_n \) satisfies the recurrence relation

\[
g_1 = 1, g_2 = 2, \text{ and } g_n = g_{n-1} + g_{n-2} + 1 \text{ for } n > 2
\]

and the formula

\[
g_n = \frac{(3 + \sqrt{5})^n (1 + \sqrt{5}) - (3 - \sqrt{5}) (1 - \sqrt{5})^n}{2^{n+1} \sqrt{5}} - 1.
\]
The last sequence is very different from all the others. Let $m_n$ be the $n$th meandric number. Then

$$m_1 = 1, m_2 = 2, m_3 = 8, m_4 = 42, m_5 = 262, m_6 = 1828, \ldots$$

As of today, there is no formula or recurrence relation to describe this sequence – but maybe you’ll figure one out!
1.1 Exercises

Write down a formula and/or a recurrence relation for the following sequences.

1. 23, 23, 23, 23, 23, 23, ...

2. 0, 1, 8, 27, 64, 125, 216, ...

3. 7, 8, 15, 34, 71, 132, 223, ...

4. 0, 1, 0, −1, 0, 1, 0, −1, ...

5. 1, 2, 6, 24, 120, 720, ...

6. 1, 7, 19, 37, 61, 91, ...
1.2 Solutions

1. 23, 23, 23, 23, 23, 23, . . .
   \[ a_n = 23; \quad a_1 = 23, a_n = a_{n-1} \text{ for } n > 1 \]

2. 0, 1, 8, 27, 64, 125, 216, . . .
   \[ b_n = n^3; \quad b_0 = 0, b_{n+1} = b_n + z_n \]

3. 7, 8, 15, 34, 71, 132, 223, . . .
   \[ c_n = b_n + 7 = n^3 + 7; \quad c_0 = 7, c_{n+1} = c_n + z_n \]

4. 0, 1, 0, −1, 0, 1, 0, −1, . . .
   \[ x_n = \sin \left( \frac{n\pi}{2} \right) \]

5. 1, 2, 6, 24, 120, 720, . . .
   \[ y_n = n! \]

6. 1, 7, 19, 37, 61, 91, . . .
   \[ z_n = b_{n+1} - b_n = 3n^2 + 3n + 1; \quad z_0 = 1, z_n = z_{n-1} + 6n \text{ for } n > 1 \]
2 The Difference Operator

How did the exercises go? My guess is that you were able to write down formulae for exercises 1 through 5 quickly, but only came up with a recurrence relation for exercise 6. What is hard about exercise 6? You have probably seen the other five sequences before (or at least sequences very similar) and as a result you already have a good intuition for the numbers appearing in these sequences. But the sequence in exercise 6 is not so common. Let’s look at it again:

1, 7, 19, 37, 61, 91, ...

Let’s call the nth term in this sequence \( z_n \) (here we will assume \( z_0 = 1, z_1 = 7 \), etc.). Is this sequence related to any of the other sequences from the exercises?

Surprisingly (or not), it is related to the sequence in exercise 2. Let \( b_n \) be this sequence. We can easily spot the formula for \( b_n \):

\[
b_n = n^3.
\]

Now what was the recurrence relation you came up with for \( b_n \)? One recurrence relation is

\[
b_0 = 0, \ b_{n+1} = b_n + z_n \quad (n \geq 0).
\]

Another way to look at this is that the difference between the \( (n+1) \)th and \( n \)th element of the sequence \( b_n \) is the \( n \)th element in the sequence \( z_n \):

\[
b_{n+1} - b_n = z_n \quad (n \geq 0).
\]

Therefore, the formula for \( z_n \) is

\[
z_n = b_{n+1} - b_n = (n + 1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1.
\]

We have a name for this relationship between the sequences \( b_n \) and \( z_n \). We say that the sequence \( z_n \) is the difference of the sequence \( b_n \) and write

\[
\Delta b_n = b_{n+1} - b_n = z_n
\]

and call \( \Delta \) the difference operator.

Now notice something else. Let \( c_n \) represent the \( n \)th element of the sequence in exercise 3. It is clear that

\[
c_n = n^3 + 7.
\]

What is \( \Delta c_n \)?

\[
\Delta c_n = c_{n+1} - c_n = (n + 1)^3 + 7 - (n^3 + 7) = n^3 + 3n^2 + 3n + 1 + 7 - n^3 - 7 = 3n^2 + 3n + 1 = z_n.
\]
Wow! The difference of $c_n$ is again $z_n$. At first this may seem surprising, but actually it is quite obvious given the relationship between $b_n$ and $c_n$.

What are the differences of the other sequences in the exercises? Let $a_n$ be the sequence in exercise 1. Then $a_n = 23$ and

$$\Delta a_n = a_{n+1} - a_n = 23 - 23 = 0.$$ 

Let $x_n$ be the sequence in exercise 4. Then $x_n = \sin(n\pi/2)$ and

$$\Delta x_n = x_{n+1} - x_n$$

$$= \sin\left(\left(n + 1\right)\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right)$$

$$= \sin\left(n\frac{\pi}{2} + \frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right)$$

$$= \sin\left(n\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) - \cos\left(n\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right)$$

$$= \sin\left(n\frac{\pi}{2}\right) \cdot 0 - \cos\left(n\frac{\pi}{2}\right) \cdot 1 - \sin\left(n\frac{\pi}{2}\right)$$

$$= -\cos\left(n\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right).$$

Using this formula to compute the first few elements of the difference sequence gives us

$$\Delta x_0 = -1, \Delta x_1 = -1, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = -1, \Delta x_5 = -1, \Delta x_6 = 1, \Delta x_7 = 1, \ldots$$

which is exactly what we expect.

Let $y_n$ represent the $n$th element in the sequence of exercise 5. Then $y_n = n!$, and

$$\Delta y_n = y_{n+1} - y_n$$

$$= (n + 1)! - n!$$

$$= (n + 1) \cdot n! - n!$$

$$= (n + 1 - 1) \cdot n!$$

$$= n \cdot n!$$

Finally, we know that $z_n = 3n^2 + 3n + 1$. Thus

$$\Delta z_n = z_{n+1} - z_n$$

$$= 3(n + 1)^2 + 3(n + 1) + 1 - (3n^2 + 3n + 1)$$

$$= 3(n^2 + 2n + 1) + 3n + 3 + 1 - 3n^2 - 3n - 1$$

$$= 3n^2 + 6n + 3n + 3 + 1 - 3n^2 - 3n - 1$$

$$= 6n + 6$$

$$= 6(n + 1)$$

Isn’t this fun???
2.1 Exercises

Compute the difference of each of the following sequences.

1. $n^4$

2. $e^n$

3. $\ln(n)$

4. $n^3 \equiv \frac{n!}{(n-3)!}$ for $n \geq 3$

5. $\binom{n}{3} \equiv \frac{n!}{(n-3)!3!}$ for $n \geq 3$
2.2 Solutions

1. \[ \Delta n^4 = (n+1)^4 - n^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 = 4n^3 + 6n^2 + 4n + 1 \]

2. \[ \Delta e^n = e^{n+1} - e^n = e \cdot e^n - e^n = (e - 1)e^n \]

3. \[ \Delta \ln(n) = \ln(n+1) - \ln(n) = \ln \left( \frac{n+1}{n} \right) = \ln \left( 1 + \frac{1}{n} \right) \approx \frac{1}{n} \]

4. \[
\Delta n^3 = (n+1)^3 - n^3 \\
= \frac{(n+1)!}{(n+1-3)!} - \frac{n!}{(n-3)!} \\
= \frac{(n+1)!}{(n-2)!} - \frac{(n-2)!}{(n-2)!} \\
= \frac{(n+1) \cdot n! - (n-2) \cdot n!}{(n-2)!} \\
= \frac{(n+1 - (n-2)) \cdot n!}{(n-2)!} \\
= \frac{3 \cdot n!}{(n-2)!} \\
= 3n^2
\]

5. \[
\Delta \left( \begin{array}{c} n \\ 3 \end{array} \right) = \left( \begin{array}{c} n+1 \\ 3 \end{array} \right) - \left( \begin{array}{c} n \\ 3 \end{array} \right) \\
= \frac{(n+1)!}{(n+1-3)!3!} - \frac{n!}{(n-3)!3!} \\
= \frac{1}{3!} \left[ \frac{(n+1)!}{(n-2)!} - \frac{n!}{(n-3)!} \right] \\
= \frac{1}{3!} \cdot \Delta n^2 \\
= \frac{1}{3!} \cdot 3 \cdot \frac{n!}{(n-2)!} \\
= \frac{n!}{(n-2)!2!} \\
= \left( \begin{array}{c} n \\ 2 \end{array} \right)
\]
3 More on the Difference Operator

This is fun, isn’t it? We now prove an important theorem about the difference operator.

**Theorem 1.** Let \( a_n \) and \( b_n \) be sequences, and let \( c \) be any number. Then

1. \( \Delta(a_n + b_n) = \Delta a_n + \Delta b_n \), and
2. \( \Delta(c \cdot a_n) = c \cdot \Delta a_n \).

**Proof.** We simply calculate

\[
\Delta(a_n + b_n) = (a_{n+1} + b_{n+1}) - (a_n + b_n) = (a_{n+1} - a_n) + (b_{n+1} - b_n) = \Delta a_n + \Delta b_n
\]

and

\[
\Delta(c \cdot a_n) = c \cdot a_{n+1} - c \cdot a_n = c \cdot (a_{n+1} - a_n) = c \cdot \Delta a_n
\]

\(\square\)

In the language of mathematics, Theorem 1 tells us that the difference operator is a linear operator. This is a very convenient property for an operator to have, as will become evident in the following proposition.

**Proposition 1 (The difference of a polynomial).** Let \( a_n \) be a polynomial in \( n \) of degree \( k \geq 1 \). Then \( \Delta a_n \) is a polynomial of degree \( k - 1 \).

We will begin by proving a short lemma.

**Lemma 1.** \( \Delta n^k = \sum_{i=0}^{k-1} \binom{k}{i} n^i \) for all \( k = 1, 2, 3, \ldots \).

**Proof.** By the binomial theorem

\[
(n + 1)^k = \sum_{i=0}^{k} \binom{k}{i} n^i
\]

\[= \binom{k}{0} + \binom{k}{1} n + \binom{k}{2} n^2 + \cdots + \binom{k}{k-1} n^{k-1} + \binom{k}{k} n^k.
\]
Notice \( \binom{k}{k} = \frac{p!}{(k-k)!n!} = 1 \) so

\[
\Delta n^k = (n + 1)^k - n^k
\]

\[
= \sum_{i=0}^{k} \binom{k}{i} n^i - n^k
\]

\[
= \sum_{i=0}^{k-1} \binom{k}{i} n^i + \binom{k}{k} n^k - n^k
\]

\[
= \sum_{i=0}^{k-1} \binom{k}{i} n^i
\]

\[
\square
\]

**Proof of Proposition 1.** If \( a_n \) is a polynomial in \( n \) of degree \( k \), then

\[
a_n = \sum_{i=0}^{k} c_i n^i = c_0 + c_1 n + c_2 n^2 + \cdots + c_{k-1} n^{k-1} + c_k n^k
\]

for some constants \( c_0, c_1, \ldots, c_k \) where \( c_k \neq 0 \). Then by the linearity of the difference operator

\[
\Delta a_n = \Delta \sum_{i=0}^{k} c_i n^i = \sum_{i=0}^{k} c_i \Delta n^i.
\]

We just saw in Lemma 1 that \( \Delta n^i \) is a polynomial of degree \( i - 1 \) for \( i \geq 1 \). For \( i = 0, \Delta n^0 = \Delta 1 = 0 \). Therefore, \( \Delta a_n \) is the sum of polynomials of degrees 0 through \( k - 1 \). Thus \( \Delta a_n \) is a polynomial of degree \( k - 1 \).

Now it is your turn to compute the difference of a few general sequences. Use the linearity of the difference operator when possible.
3.1 Exercises

Compute the difference of the following sequences. Here \( b > 0 \), \( c \) is a constant and \( k \) is a natural number such that \( 1 \leq k \leq n \).

1. \( c \cdot b^n \) (exponential sequence)

2. \( c \cdot \log_b(n) \) (logarithmic sequence)

3. \[ n^k = \frac{n!}{(n-k)!} \] (falling factorial sequence)

4. \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \) (binomial coefficient sequence)
3.2 Solutions

1. \[ \Delta c \cdot b^n = c \Delta b^n \]
\[ = c(b^{n+1} - b^n) \]
\[ = c(b \cdot b^n - b^n) \]
\[ = c(b - 1)b^n \]

2. \[ \Delta c \log_b(n) = c \Delta \log_b(n) \]
\[ = c (\log_b(n + 1) - \log_b(n)) \]
\[ = c \log_b \left( \frac{n + 1}{n} \right) \]
\[ = c \log_b \left( 1 + \frac{1}{n} \right) \]

3. \[ \Delta n^k = (n + 1)^k - n^k \]
\[ = \frac{(n + 1)!}{(n + 1 - k)!} - \frac{n!}{(n - k)!} \]
\[ = \frac{(n + 1)!}{(n + 1 - k)!} - \frac{(n + 1 - k) \cdot n!}{(n + 1 - k) \cdot (n - k)!} \]
\[ = \frac{(n + 1) \cdot n! - (n + 1 - k) \cdot n!}{(n + 1 - k) \cdot (n - k)!} \]
\[ = \frac{k \cdot n!}{(n + 1 - k)!} \]
\[ = k \frac{n!}{(n - (k - 1))!} \]
\[ = k n^{k-1} \]
4. This follows easily from the previous exercise since \(\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{1}{k!} n^k\):

\[
\Delta \left( \binom{n}{k} \right) = \frac{1}{k!} n^k
\]

\[
= \frac{1}{k!} \Delta n^k
\]

\[
= \frac{k}{k!} n^{k-1}
\]

\[
= \frac{1}{(k-1)!} \frac{n!}{(n-(k-1))!}
\]

\[
= \frac{n!}{(n-(k-1))!(k-1)!}
\]

\[
= \binom{n}{k-1}
\]

One could also see this by considering Pascal’s triangle.
3.3 Take-Home Exercises

For the first three exercises, it is helpful to recall that

\[ n^k = \frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-k+2)(n-k+1) \]

That is, \( n^k \) is a polynomial in \( n \) of degree \( k \).

1. Give the formula for a sequence \( a_n \) such that \( \Delta a_n = n \).
   (hint: notice \( n = n^1 \) and recall that \( \Delta n^2 = 2n^1 \))

2. Give the formula for a sequence \( b_n \) such that \( \Delta b_n = n^2 \).
   (hint: notice \( n^2 = n^2 + n^1 \))
3. Give the formula for a sequence $c_n$ such that $\Delta c_n = n^3$.
   (hint: write $n^3$ as a linear combination of $n^{\frac{1}{3}}$, $n^{\frac{2}{3}}$, and $n^{\frac{1}{3}}$)

4. Give the formula for a sequence $d_n$ such that $\Delta d_n = d_n$.
   (hint: consider an exponential sequence)
3.4 Solutions to Take-Home Exercises

1. \( a_n = \frac{1}{2}n^2 = \frac{n(n-1)}{2} \), for example

2. \( b_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 = \frac{n(n-1)(2n-1)}{6} \), for example

3. \( c_n = \frac{1}{4}n^4 + n^2 + \frac{1}{2}n^2 = \frac{n^2(n-1)^2}{4} \), for example

4. \( d_n = 2^n \), for example