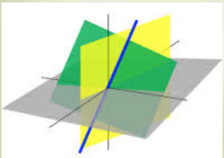
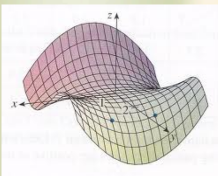
Tangent Planes

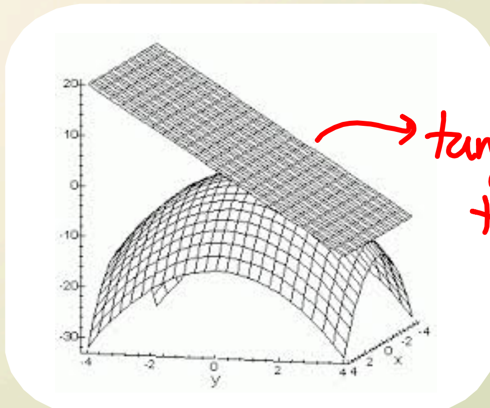


$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$



$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y}dy$$
$$= \int_0^1 \frac{(2y)^2}{2}y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$



tangent plane to the surface

Tangent Planes

We already dealt with tangent planes to surfaces of form $z = f(x,y)$. Now, we will find tangent planes to surfaces of form $\boxed{F(x,y,z) = k,}$ i.e. a surface represented by any equation in three variables.

Definition

Let $F(x,y,z)=k$ be a surface, F, differentiable at $P(x_0,y_0,z_0)$ with $\nabla F(x_0,y_0,z_0) \neq \vec{0}$. Then the plane through P and perpendicular to $\nabla F(x_0,y_0,z_0)$ is called the tangent plane to the surface at P.

Theorem

For surface $F(x,y,z)=k$, the equation of the tangent plane at (x_0,y_0,z_0) is

$$\boxed{\underbrace{\nabla F(x,y,z)}_{\text{3-d gradient}} \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0}$$

$\Leftrightarrow$

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

EX 1 Find the equation of the tangent plane to $\underbrace{8x^2+y^2+8z^2}_{F(x,y,z)}=16$ at $\left(1, 2, \frac{\sqrt{2}}{2}\right)$.

$\nabla F = \langle F_x, F_y, F_z\rangle = \langle 16x, 2y, 16z\rangle$

$\vec{n} = \nabla F\left(1,2,\frac{\sqrt{2}}{2}\right) = \langle 16(1), 2(2), 16\left(\frac{\sqrt{2}}{2}\right)\rangle$

$= \langle 16, 4, 8\sqrt{2}\rangle = 4\langle 4, 1, 2\sqrt{2}\rangle$

pt

$(x_0, y_0, z_0) = \left(1, 2, \frac{\sqrt{2}}{2}\right)$

$\nabla F\left(1,2,\frac{\sqrt{2}}{2}\right) \cdot \langle x-1, y-2, z-\frac{\sqrt{2}}{2}\rangle = 0$

$\langle 4, 1, 2\sqrt{2}\rangle \cdot \langle x-1, y-2, z-\frac{\sqrt{2}}{2}\rangle = 0$

$4x-4+y-2+2\sqrt{2}z-2=0$

$$\boxed{4x+y+2\sqrt{2}z=8}$$

$z = f(x,y)$ (explicitly given fn)

$\underbrace{z - f(x,y)}_{F(x,y,z)} = 0$

$F_x = -f_x$, $F_y = -f_y$

$F_z = 1$

tangent plane at (x_0, y_0, z_0):

$\langle F_x, F_y, F_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\langle -f_x, -f_y, 1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$-f_x(x - x_0) - f_y(y - y_0) + z - z_0 = 0$

$$\boxed{z = z_0 + f_x(x - x_0) + f_y(y - y_0)}$$

eqn of tangent plane from the "past"

EX 2 Find parametric equations of the line that is tangent to the curve of intersection of these surfaces at the point $(1,2,2)$.

$$f(x,y,z) = 9x^2 + 4y^2 + 4z^2 - 41=0$$
$$g(x,y,z) =2x^2 - y^2 + 3z^2 - 10 = 0$$

For a line, we need a vector $\vec{v}$ in direction of the line and we need a pt on the line. $(1,2,2)$

The line that's tangent to curve of intersectn is $\perp$ to ∇f and ∇g.

$$\Rightarrow \vec{v} = \nabla f \times \nabla g$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 18x, 8y, 8z \rangle$$

$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 4x, -2y, 6z \rangle$$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 18x & 8y & 8z \\ 4x & -2y & 6z \end{vmatrix} = (48yz + 16yz)\hat{i} - (108xz - 32xz)\hat{j} + (-36xy - 32xy)\hat{k}$$

$$= 64yz\hat{i} - 76xz\hat{j} - 68xy\hat{k}$$

$$= 4(16yz\hat{i} - 19xz\hat{j} - 17xy\hat{k})$$

at $(1,2,2)$

$$\vec{v} = 4(16(2)(2)\hat{i} - 19(1)(2)\hat{j} - 17(1)(2)\hat{k})$$

$$= 4(2)\underbrace{(32\hat{i} - 19\hat{j} - 17\hat{k})}$$

tangent line:

$$\begin{cases} x = 1 + 32t \\ y = 2 + (-19t) \\ z = 2 + (-17t) \end{cases}$$

Definition ← z as explicit fn of x and y

Let $z = f(x,y)$, f is differentiable function, dx and dy (differentials) are variables. dz (also called total differential of f) is

$$dz = df(x,y) = f_x(x,y)dx + f_y(x,y)dy = \nabla f \cdot \langle dx, dy \rangle.$$

dx = "a little bit of x"

dy = "a little bit of y"

remainder: tangent plane: (for explicit surface) at input pt (a,b)

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\underbrace{z - f(a,b)}_{dz} = f_x(a,b)dx + f_y(a,b)dy \quad \text{if } (a,b) \approx (x,y)$$

$$dz = f_x dx + f_y dy$$

(i.e. tangent plane approximates surface well if we don't go too far away from (a,b))

EX 3 Use differentials to approximate the change in z as (x,y) moves from P to Q. Also find Δz. (actual change)

$z = \underbrace{x^2 - 5xy + y}_{f}$ $\quad P(2,3) \quad Q(2.03, 2.98)$

$$dz = f_x dx + f_y dy$$

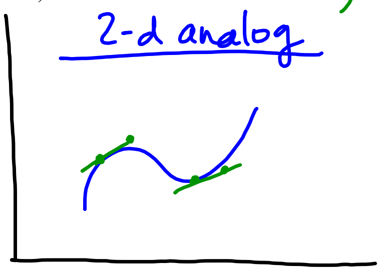
$$dx = 2.03 - 2 = 0.03$$

$$dy = 2.98 - 3 = -0.02$$

$f_x = 2x - 5y$ at pt $(2,3)$

$f_y = -5x + 1$

2-d analog



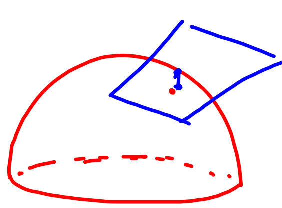
$$f_x = 2(2) - 5(3) = 4 - 15 = -11$$

$$f_y = -5(2) + 1 = -9$$

$$dz = -11(0.03) + -9(-0.02)$$
$$= -0.33 + 0.18 = -0.15$$ (approx change in z)

actual change in z:

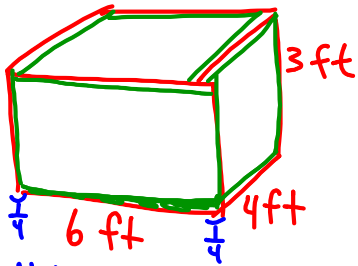
$$\Delta z = z(2.03, 2.98) - z(2,3)$$
$$= (2.03^2 - 5(2.03)(2.98) + 2.98) - (2^2 - 5(2)(3) + 3)$$
$$= -23.1461 - (-23)$$
$$= -0.1461$$



EX 4 Use differentials to find the approximate amount of copper in the four sides and bottom of a rectangular copper tank that is 6 feet long, 4 feet wide and 3 feet deep inside, if the sheet-copper is 1/4 inch thick.

(note: 3, 6, 4 ft measurements are outer measurements)

(rectangular box w/ <u>no</u> lid)



we want to approximate ΔV.

$$V(x,y,z) = xyz$$

$$dV \approx \Delta V$$

$$dV = V_x\,dx + V_y\,dy + V_z\,dz = yz\,dx + xz\,dy + xy\,dz$$

$dx = dy = dz = \frac{1}{2}$ in $= 0.5$ in, $\quad x = 4$ ft, $y = 6$ ft, $z = 3$ ft

$$dx = dy = dz = \frac{1}{24} \text{ ft}$$

$$\Delta V \approx dV = 6(3)\left(\tfrac{1}{24}\right) + 4(3)\left(\tfrac{1}{24}\right) + 4(6)\left(\tfrac{1}{24}\right)$$

$$= \frac{3}{4} + \frac{1}{2} + 1 = 2\tfrac{1}{4} \text{ or } \boxed{\frac{9}{4} \text{ ft}^3}$$

(approx volume of sheet copper)

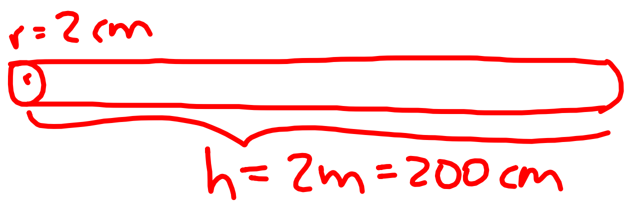
<u>note</u>: actual volume of sheet-copper

$$\Delta V = V_{outside} - V_{inside}$$

relative volume: $\dfrac{\text{approx Vol. of sheet copper}}{\text{Vol. of tank}}$

$$= \frac{9/4}{3(6)(4)} = \frac{1}{32} = 0.03125 = 3.125\%$$

EX 5 A piece of cable (cylindrical) that measures 2 meters long with a radius of 2 centimeters is thought to have measurement error as large as 5 millimeters for each of the height and radius measurements. Estimate the error in the volume measurement.



$dh = dr = 5\text{ mm}$

$= 0.5\text{ cm}$

$V = \pi r^2 h$

$$\Delta V \approx dV = V_r\,dr + V_h\,dh$$

$$= 2\pi r h\,dr + \pi r^2\,dh$$

$$= 2\pi(2)(200)(0.5) + \pi(2^2)(0.5)$$

$$= 400\pi + 2\pi = 402\pi \simeq 1262.9\text{ cm}^3$$

approx error in volume is $\pm$ 1262.9 cm^3

for explicit fn:

$f(x,y)=z$

$\nabla f = \langle f_x, f_y \rangle$

$\perp$ to level curves of $z=f(x,y)$

for implicit fn:

$F(x,y,z)=k$, $k \in \mathbb{R}$ (constant)

$\nabla F = \langle F_x, F_y, F_z \rangle$

$\perp$ to surface!!!

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for explicit fn $z=f(x,y)$

$\iff \underbrace{f(x,y)-z}_{F}=0$

3d gradient $\langle f_x, f_y, -1 \rangle$

$\perp$ to surface $z=f(x,y)$.