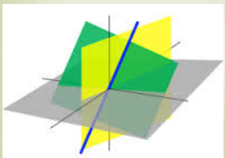
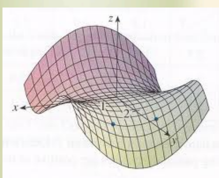
The Chain Rule



$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

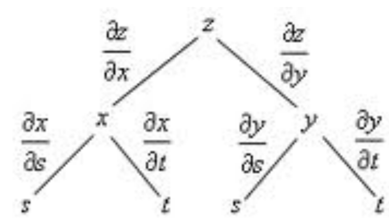
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2} \right]_{y=0}^{y=1} = \frac{1}{2}$$



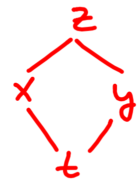
Recall: Chain rule for $y = f(g(x))$ is $y' = f'(g(x))g'(x) = \frac{df}{dg}\frac{dg}{dx}$.

Chain Rules

Theorem

Let $x = x(t)$ and $y = y(t)$ be differentiable at t and let $z = f(x,y)$ be differentiable at $(x(t),y(t))$.
Then $z = f(x(t),y(t))$ is differentiable at t and

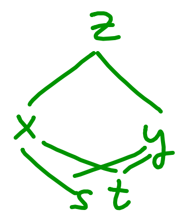
$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}} \Leftrightarrow \underbrace{\frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}$$



Theorem

Let $x = x(s,t)$ and $y = y(s,t)$ have first partial derivatives at $= \langle s, t \rangle$ and let $z = f(x,y)$ be differentiable at $(x(s,t), y(s,t))$.
Then z has first partial derivatives given by

$$\boxed{\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\end{aligned}}$$



EX 1 Find $\frac{\partial w}{\partial t}$ given $w = x^2y - y^2x$, $x = \cos t$, $y = \sin t$.
Express the answer in terms of t.

① $w = w(x,y)$

$w = x^2y - y^2x$

$= (\cos t)^2(\sin t) - (\sin t)^2 \cos t$

$= \cos^2 t \sin t - \sin^2 t \cos t$

$\frac{dw}{dt} = 2\cos t(-\sin t)\sin t + \cos^2 t(\cos t) - \left(2\sin t\cos t(\cos t) + \sin^2 t(-\sin t)\right)$

$\frac{dw}{dt} = -2\cos t\sin^2 t + \cos^3 t - 2\sin t\cos^2 t + \sin^3 t$

② $w(x,y) = x^2y - y^2x$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$= (2xy - y^2)(-\sin t) + (x^2 - 2yx)(\cos t)$

$= (2\sin t\cos t - \sin^2 t)(-\sin t) + (\cos^2 t - 2\cos t\sin t)\cos t$

EX 2 Find $\frac{\partial w}{\partial t}$ given $w = ln(x+y) - ln(x-y), \quad x = te^{s}, \quad y = e^{st}.$

Express the answer in s and t .

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \left(\frac{1}{x+y} - \frac{1}{x-y}\right)\left(e^{s}\right) + \left(\frac{1}{x+y} - \frac{-1}{x-y}\right)\left(e^{st}s\right)$$

$$= \left(\frac{1}{te^{s}+e^{st}} - \frac{1}{te^{s}-e^{st}}\right)\left(e^{s}\right) + \left(\frac{1}{te^{s}+e^{st}} + \frac{1}{te^{s}-e^{st}}\right)\left(e^{st}s\right)$$

EX 3 If $w = xy + x + y, \quad x = r + s + t \quad and\ y = rst,$

find $\left.\frac{\partial w}{\partial t}\right|_{r=1,\, s=-1,\, t=2}$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial t} = (y+1)(1) + (x+1)(rs)$$

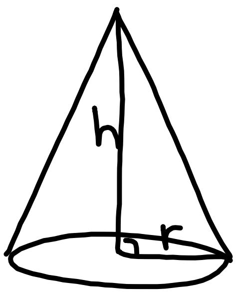
$$= rst+1 + rs(r+s+t+1)$$

$$\left.\frac{\partial w}{\partial t}\right|_{(1,-1,2)=(r,s,t)} = 1(-1)(2)+1+1(-1)(1+-1+2+1)$$

$$= -2+1-(3)$$

$$= -4$$

EX 4 Sand is pouring onto a conical pile in such a way that at a certain instant, the height is 100 inches and increasing at 3 in/min. The base radius at that instant is 40 inches and increasing at 2 in/min. How fast is the volume increasing at that instant?



r, h are changing over time

$$V = \frac{1}{3}\pi r^2 h \qquad (V = V(r,h))$$

know: $\begin{cases} h = 100\text{ in} & \frac{dh}{dt} = 3\text{ in/min} \\ r = 40\text{ in} & \frac{dr}{dt} = 2\text{ in/min.} \end{cases}$

Qn: $\frac{dV}{dt} = ?$ when above is true

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi r h\left(\frac{dr}{dt}\right) + \frac{1}{3}\pi r^2\left(\frac{dh}{dt}\right)$$

$$= \frac{2}{3}\pi(40)(100)(2) + \frac{1}{3}\pi(40^2)(3)$$

$$= \frac{16000}{3}\pi + 1600\pi = \frac{16000 + 4800}{3}\pi$$

$$= \frac{20800}{3}\pi \ \text{in}^3/\text{min}$$

$$\simeq 21{,}781.71 \ \text{in}^3/\text{min}$$

Implicit Differentiation

Let's go back to $y = f(x)$ and assume that instead of getting y as a function of x (explicitly), we have $F(x,y) = k$, for any constant, k (i.e. y is defined implicitly). Then, we just differentiated both sides with respect to x to get dy/dx .

y is a fn of x.

For example:

$$\frac{d}{dx}\left(y^3 - 2xy + 3x = 4\right)$$

$$\frac{d}{dx}(y^3) - 2\frac{d}{dx}(xy) + 3\frac{d}{dx}(x) = \frac{d}{dx}(4)$$

$$3y^2\frac{dy}{dx} - 2\left(1y + x\frac{dy}{dx}\right) + 3(1) = 0$$

$$\frac{dy}{dx}(3y^2 - 2x) = 2y - 3$$

$$\boxed{\frac{dy}{dx} = \frac{2y-3}{3y^2-2x}}$$

$z = z(x,y)$

Apply the same ideas:

$$xy^3 + sec(y+z) - zx^2 = 1$$

$$\frac{\partial}{\partial x}\left(xy^3 + \sec(y+z) - zx^2\right) = \frac{\partial}{\partial x}(1)$$

$$y^3 + \sec(y+z)\tan(y+z)\left(\frac{\partial z}{\partial x}\right) - \left(\frac{\partial z}{\partial x}x^2 + z(2x)\right) = 0$$

$$\frac{\partial z}{\partial x}\left(\sec(y+z)\tan(y+z) - x^2\right) = -y^3 + 2xz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-y^3 + 2xz}{\sec(y+z)\tan(y+z) - x^2}}$$

we can similarly find $\frac{\partial z}{\partial y}$.

EX 5 If $ye^{-x} + \sin(x+z) + e^z = 5$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

① $\frac{\partial z}{\partial x}$: (y does not depend on x, but z does depend on x)

$$\frac{\partial}{\partial x}\left(ye^{-x} + \sin(x+z) + e^z\right) = \frac{\partial}{\partial x}(5)$$

$$-ye^{-x} + \cos(x+z)\left(1 + \frac{\partial z}{\partial x}\right) + e^z\left(\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial z}{\partial x}\left(\cos(x+z) + e^z\right) = ye^{-x} - \cos(x+z)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{ye^{-x} - \cos(x+z)}{\cos(x+z) + e^z}}$$

② $\frac{\partial z}{\partial y}$: (x is not dependent on y, but z does depend on y)

$$\frac{\partial}{\partial y}\left(ye^{-x} + \sin(x+z) + e^z\right) = \frac{\partial}{\partial y}(5)$$

$$e^{-x} + \cos(x+z)\left(\frac{\partial z}{\partial y}\right) + e^z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}\left(\cos(x+z) + e^z\right) = -e^{-x}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-e^{-x}}{\cos(x+z) + e^z}}$$