If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

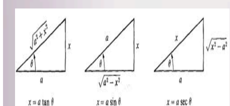
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

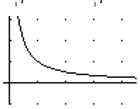
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

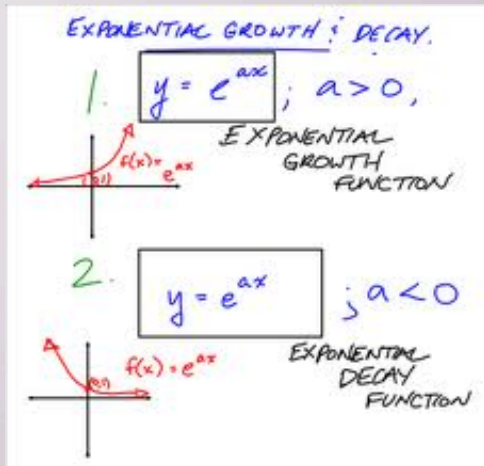
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Exponential Growth and Decay



Population Growth

y = population
t = time

Let y_0 = population at the start.
$dy/dt = ky$ is a reasonable assumption.

I.e. $\underbrace{\text{The rate of growth}}_{\frac{dy}{dt}}$ or decay is proportional to the population.

$$\frac{dy}{dt} = ky$$

y ← population

k ↑ proportionality constant

$$\frac{dy}{dt} = ky$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$(y>0)$ $\ln|y| = kt + C$

$$\ln y = kt + c$$

$$e^{\ln y} = e^{kt+c}$$

$$y = e^{kt}e^{c}$$

$$y = e^{c}(e^{kt})$$

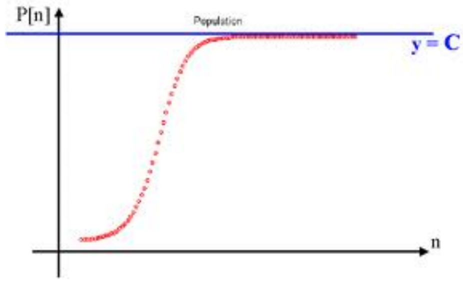
at $t=0$, $y=y_0$.

$$\Rightarrow y_0 = e^{c}(e^{k\cdot 0})$$

$$\Rightarrow y_0 = e^{c} \Rightarrow \boxed{y = y_0 e^{kt}}$$

familiar eqn to model population growth
(or some exp. growth/decay)

There are other factors that affect population, like resources and land. So population more likely follows a logistic model like this.



EX 1 The population of the US was 3.9 million in 1790 and 178 million in 1960. If the rate of growth is assumed to be proportional to the population, what estimate would you give for the population in the year 2000? (actual answer = 275 million)

$y = y_0 e^{kt}$

Let $t = 0$ in year 1790 $\Rightarrow y_0 = 3.9$ (million)

exponential growth curve goes through

$(0, 3.9)$ $(170, 178)$ goal: $(210, ?)$

$y = 3.9e^{kt}$

plug in $(170, 178)$ $178 = 3.9e^{k(170)}$

$$\frac{178}{3.9} = e^{170k}$$

$$\frac{1}{170}\ln\left(\frac{178}{3.9}\right) = k$$

goal: $y = 3.9\, e^{\left(\frac{1}{170}\ln\left(\frac{178}{3.9}\right)\right)(210)} \approx \boxed{437.4 \text{ million}}$

Compound Interest Formula

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

A_0 = initial amount
A(t) = value after t years
r = Interest rate
n = number of compounding periods (per year)

What if we compound continuously?

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

Theorem

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e \qquad (1^{\infty} \text{ case})$$

Proof

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$

$$\Leftrightarrow \quad 1 = f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \qquad \text{(by defn of derivative)}$$

$$1 = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} \qquad (\ln 1 = 0)$$

$$1 = \lim_{h\to 0} \frac{\ln(1+h)}{h}$$

$$1 = \lim_{h\to 0} \frac{1}{h}\ln(1+h) \Leftrightarrow 1 = \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}}$$

$$1 = \ln\left(\lim_{h\to 0}(1+h)^{\frac{1}{h}}\right)$$

(note: this can be done because ln fn is continuous)

$$e = e^1 = \lim_{h\to 0}(1+h)^{\frac{1}{h}} \quad \#$$

Continuous compounding:

EX 2 Compute this limit to get a formula for continuously compounded interest.

$$A(t)=\lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$A = A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t}$$

$$= A_0\left(\lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{n}\right)^{t}$$

$$= A_0\left(\lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{n/r}\right)^{rt}$$

$$(a^{n/r})^{r}=a^{n}$$

$$(a^{n/r})^{rt}=a^{nt}$$

$$= A_0\left(\lim_{n/r\to\infty}\left(1+\frac{1}{n/r}\right)^{n/r}\right)^{rt}$$

$$= A_0\left(\lim_{p\to\infty}\left(1+\frac{1}{p}\right)^{p}\right)^{rt}$$

$$\left(\text{let } p=\frac{n}{r}\right)$$

$$= A_0\left(\underline{\lim_{h\to 0}(1+h)^{\frac{1}{h}}}\right)^{rt}$$

let $h=\frac{1}{p}$ $\left(\frac{1}{h}=p\right)$

we have

$$e=\lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

$$|||$$

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

$$A=A_0(e)^{rt}$$

$$\boxed{A=A_0e^{rt}}$$

exponential growth

EX 3 Compute this limit.

$$\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^x = e^2$$

use

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}} \quad ①$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad ②$$

do long division:

$$\begin{array}{r} 1+\frac{2}{x+1} \\ x+1\overline{)\,x+3} \\ -(x+1) \\ \hline 2 \end{array}$$

$$\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^x = \lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^x$$

$$= \lim_{x\to\infty}\frac{\left(1+\frac{1}{\frac{(x+1)}{2}}\right)^{x+1}}{\left(1+\frac{1}{\frac{(x+1)}{2}}\right)}$$

$$a^x = \frac{a^{x+1}}{a}$$

$$a^x = \left(a^{\frac{x}{2}}\right)^2$$

$$= \lim_{x\to\infty}\frac{\left(\left(1+\frac{1}{\frac{(x+1)}{2}}\right)^{\frac{x+1}{2}}\right)^2}{\left(1+\frac{1}{\frac{x+1}{2}}\right)}$$

$$= \frac{\left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x+1}{2}}\right)^{\frac{x+1}{2}}\right)^2}{\lim_{x\to\infty}\left(1+\underbrace{\frac{1}{\frac{x+1}{2}}}_{0}\right)}$$

note: if $x\to\infty$, then $\frac{x+1}{2}\to\infty$

$$= \frac{e^2}{1} = e^2$$

"take-home message"

① $\lim_{h \to 0} (1+h)^{1/h} = e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

② exponential growth/decay $y = y_0 e^{kt}$