$5x-2y \leq 75$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$S = Pe^{rt}$

$APY = (1+\frac{r}{n})^n - 1$

Math 1090 ~ Business Algebra

Section 2.4 Inverse Matrices

Objectives:

- Use Gauss-Jordan techniques to find an inverse of a matrix, if it exists.
- Use inverse matrices to solve systems of equations.

Inverse Matrix

A^{-1}, read "A inverse," is a matrix such that

- $A^{-1} \cdot A = I = A \cdot A^{-1}$ (I = identity matrix that's same size as A)
- A^{-1} can only exist for a square matrix

Ex 1: Find A^{-1} for

Method to find A^{-1} for A

1. If A is not square, A^{-1} does not exist (DNE).
2. If A is square,

   a) Augment A with the identity matrix.

   b) Perform elementary row operations on the augmented matrix until the left side is I, the identity matrix.

   c) What is on the right side is A^{-1}.

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{array}{c} \\ (-3) \\ \\ \end{array} \begin{array}{cccc} -3 & -6 & -3 & 0 \end{array}$$

$$(-3)\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\left(-\tfrac{1}{2}\right)\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$(-2)\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

$$\begin{array}{cccc} 0 & -2 & -3 & 1 \end{array}$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

b) $A = \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{5(2)-4(0)} \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Formula for A^{-1} of a 2x2 matrix (if A^{-1} exists)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

check:

$$\underset{(2\times 2)(2\times 2)}{AA^{-1}} = \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5(\frac{1}{5})+4(0) & 5(-\frac{2}{5})+4(\frac{1}{2}) \\ 0(\frac{1}{5})+2(0) & 0(-\frac{2}{5})+2(\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark = I$$

Ex 2: Find A^{-1} if possible.

a) $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -5 & 1 & 0 & 1 \\ 3 & -2 & 7 & 0 \end{bmatrix}$

3x4 matrix which is NOT square

$\Rightarrow A^{-1}$ DNE

b) $A = \begin{bmatrix} 7 & -4 & 6 \\ 7 & -4 & 5 \\ 2 & -1 & 1 \end{bmatrix}$ (3x3 matrix)

$$(-3)\quad \left[\begin{array}{ccc|ccc} 7 & -4 & 6 & 1 & 0 & 0 \\ 7 & -4 & 5 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\begin{array}{cccccc} -6 & 3 & -3 & 0 & 0 & -3 \end{array}$$

$$\begin{array}{cccccc} -7 & 7 & -21 & -7 & 0 & 21 \end{array}$$
$$(-7)\quad \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & -3 \\ 7 & -4 & 5 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} -2 & 2 & -6 & -2 & 0 & 6 \end{array}$$
$$(-2)\quad \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & -3 \\ 0 & 3 & -16 & -7 & 1 & 21 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & -3 \\ 0 & 3 & -16 & -7 & 1 & 21 \\ 0 & 1 & -5 & -2 & 0 & 7 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & -3 & 15 & 6 & 0 & -21 \end{array}$$
$$(-3)\quad \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & -3 \\ 0 & 1 & -5 & -2 & 0 & 7 \\ 0 & 3 & -16 & -7 & 1 & 21 \end{array}\right]$$

$$(-1)\quad \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & -3 \\ 0 & 1 & -5 & -2 & 0 & 7 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & -3 \\ 0 & 1 & -5 & -2 & 0 & 7 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$(5)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 4 \\ 0 & 1 & -5 & -2 & 0 & 7 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$
$$\begin{array}{cccccc} 0 & 0 & 5 & 5 & -5 & 0 \end{array}$$

$$(2)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 4 \\ 0 & 1 & 0 & 3 & -5 & 7 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$
$$\begin{array}{cccccc} 0 & 0 & 2 & 2 & -2 & 0 \end{array}$$

$$\left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{\text{3x3 identity matrix}} \middle| \underbrace{\begin{array}{ccc} 1 & -2 & 4 \\ 3 & -5 & 7 \\ 1 & -1 & 0 \end{array}}_{A^{-1}}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -5 & 7 \\ 1 & -1 & 0 \end{bmatrix}$$

★ Note: If, in this process, you get a 0 0 0 row on left side, it means A^{-1} DNE.

Ex 3: Use A^{-1} from Example 2(b) to solve this system of equations.

$$7x - 4y + 6z = 1$$
$$7x - 4y + 5z = 0$$
$$2x - y + z = 7$$

$$\left[\begin{array}{ccc|c} 7 & -4 & 6 & 1 \\ 7 & -4 & 5 & 0 \\ 2 & -1 & 1 & 7 \end{array}\right]$$

$\underbrace{}_{A}$ (from 2b)

$$A^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -5 & 7 \\ 1 & -1 & 0 \end{bmatrix}$$

To solve

$AX=B$

(where A is an *nxn* matrix

X is an *nx1* column vector of variables

B is an *nx1* column vector of constants)

we can left-multiply both sides by A^{-1}. $AX=B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$.

$X = A^{-1}B$

$AX=B$ (a matrix eqn)

$$\begin{bmatrix} 7 & -4 & 6 \\ 7 & -4 & 5 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$$

$X = A^{-1}B$

$$X = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -5 & 7 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 1(1) + -2(0) + 4(7) \\ 3(1) + -5(0) + 7(7) \\ 1(1) + -1(0) + 0(7) \end{bmatrix}$$

$3 \times 3 \quad 3 \times 1$

$$= \begin{bmatrix} 1+0+28 \\ 3+0+49 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 29 \\ 52 \\ 1 \end{bmatrix}$$

another example:

Solve
$$7x - 4y + 6z = 5$$
$$7x - 4y + 5z = 1$$
$$2x - y + z = -1$$

$$\Rightarrow B = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

(note: A is same as above $\Rightarrow A^{-1}$ is same also.)

$X = A^{-1}B$

$$= \begin{bmatrix} 1 & -2 & 4 \\ 3 & -5 & 7 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 - 2 - 4 \\ 15 - 5 - 7 \\ 5 - 1 + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$