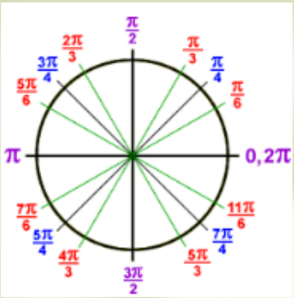
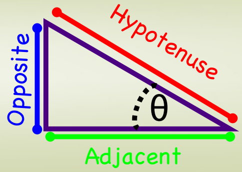
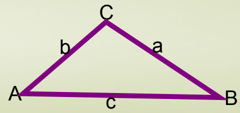
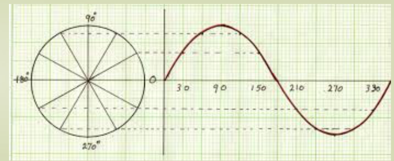








$\sin^2 u + \cos^2 u = 1$

$\sin 2u = 2 \sin u \cos u$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

Math 1060 ~ Trigonometry

12 Inverse Trigonometric Functions

Learning Objectives

In this section you will:

- **Learn and be able to apply properties of the inverse trigonometric functions, including domain and range.**
- **Find the exact values of inverse trigonometric functions.**
- **Convert compositions of trigonometric and inverse trigonometric functions to algebraic expressions.**

To find the inverse of the trigonometric functions, our first problem is that they are not one-to-one.

(i.e. $y = \sin x$ does not pass horizontal line test)

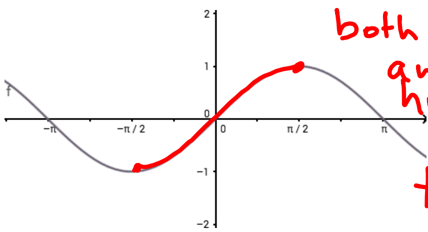
two equivalent notations

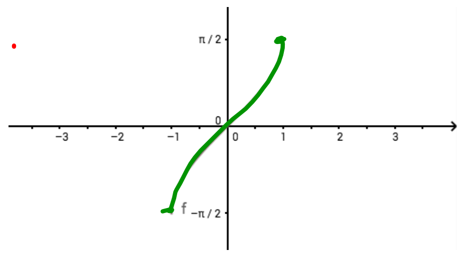
$y = \sin(x)$

the restricted piece of $y = \sin x$ curve passes both vert. and horiz. line test.

$y = \sin^{-1}(x) = \arcsin(x)$

"sine inverse of x"





restricted domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

range: $[-1, 1]$

symmetry: across origin (it's an odd fn)

domain: $[-1, 1]$

range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

symmetry: across origin (it's also an odd fn)

① does $\sin(\sin^{-1}(x)) = x$?

② does $\sin^{-1}(\sin(x)) = x$?

i.e. do sine and inverse sine always undo each other? No! (They undo each other sometimes.)

① $\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ works for all domain x-values

$\Rightarrow \sin(\sin^{-1}x) = x$ ✓

② $\sin x \in [-1, 1]$ but $\sin^{-1}(\sin x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin^{-1}(\sin x) = x$

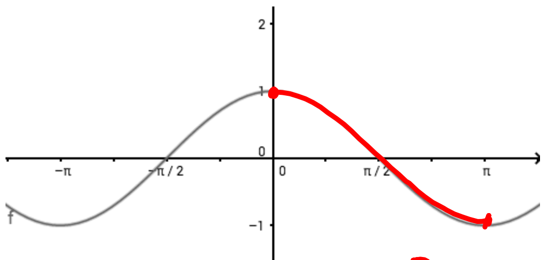
but if $x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin^{-1}(\sin x) \neq x$

remember

$\in$ means element of

$\notin$ " not an element of

$y = \cos(x)$



restricted domain: $[0, \pi]$

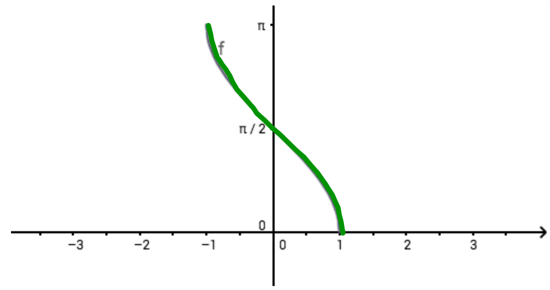
range: $[-1, 1]$

symmetry: none

① does $\cos(\cos^{-1}(x)) = x$?

Yes

$y = \cos^{-1}(x) = \arccos(x)$



domain: $[-1, 1]$

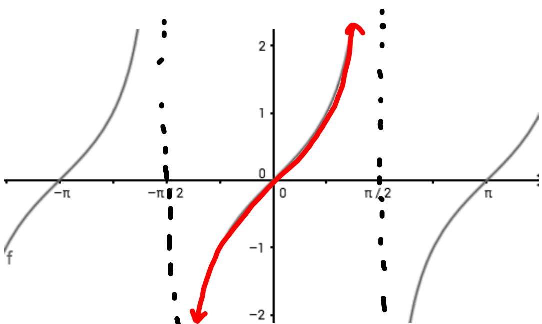
range: $[0, \pi]$

symmetry: none

② does $\cos^{-1}(\cos(x)) = x$?

No, only works sometimes, when $x \in [0, \pi]$.

$y = \tan(x)$



restricted domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

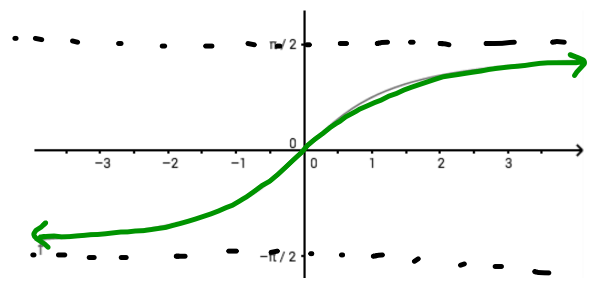
range: $(-\infty, \infty)$

symmetry: odd fn (sym. about the origin)

① does $\tan(\tan^{-1}(x)) = x$?

Yes, always works.

$y = \tan^{-1}(x) = \arctan(x)$



domain: $(-\infty, \infty)$

range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

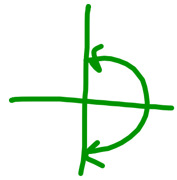
symmetry: also odd fn (sym. about the origin)

② does $\tan^{-1}(\tan(x)) = x$?

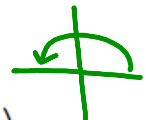
No, only works when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

When working these problems, it is easier if you think of the Unit Circle rather than the Cartesian graph.

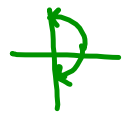
$\sin^{-1}(x)$ answers will be in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\cos^{-1}(x)$ answers will be in the interval $[0, \pi]$



$\tan^{-1}(x)$ answers will be in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Note that to compute the $\sec^{-1}(x)$, $\csc^{-1}(x)$ and $\cot^{-1}(x)$ you can turn each into a problem involving the three functions above.

ex $\theta = \sec^{-1} x$

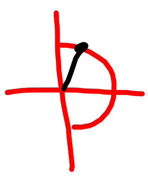
$\sec\theta = x$

$\cos\theta = \frac{1}{x}$ $\rightarrow$ $\theta = \cos^{-1}\left(\frac{1}{x}\right)$

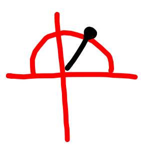
$$\boxed{\begin{aligned} \sec^{-1} x &= \cos^{-1}\left(\tfrac{1}{x}\right) \\ \csc^{-1} x &= \sin^{-1}\left(\tfrac{1}{x}\right) \\ \cot^{-1} x &= \tan^{-1}\left(\tfrac{1}{x}\right), x > 0 \end{aligned}}$$

Ex 1: Look at a Unit Circle and practice by finding the answers to these:

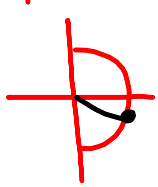
a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$



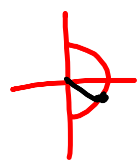
b) $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$



c) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

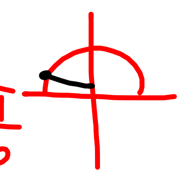


d) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



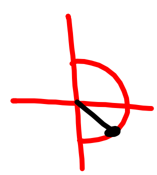
e) $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

$= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$



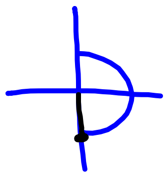
f) $\tan^{-1}(-1)$

$= -\frac{\pi}{4}$



Ex 2: Try these without looking at a Unit Circle.

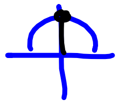
a) $\sin^{-1}(-1) = -\frac{\pi}{2}$



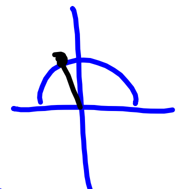
d) $\csc^{-1}(0) = \sin^{-1}\left(\frac{1}{0}\right)$ hmmm this is a problem!

undefined

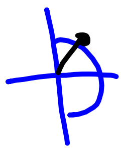
b) $\cos^{-1}(0) = \frac{\pi}{2}$



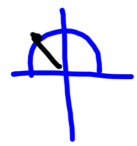
e) $\sec^{-1}(-2) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$



c) $\tan^{-1}(1) = \frac{\pi}{4}$

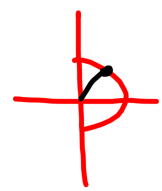


f) $\cot^{-1}(-1) = \frac{3\pi}{4}$

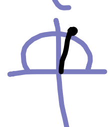


Ex 3: Which of these are true? Correct any that are false.

a) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{3\pi}{4}$ false $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} + \frac{3\pi}{4}$

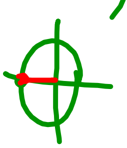


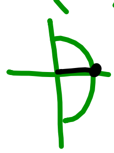
b) $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}$ true $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ ✓



c) $\tan^{-1}(\tan\pi) = \pi$ false

$\tan^{-1}(\tan\pi) = \tan^{-1}(0) = 0 \neq \pi$





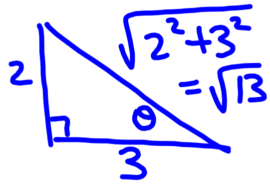
Ex 4: These will require a bit more thought and perhaps a drawing of a triangle.

Evaluate these.

a) $\cos\left(\underbrace{\arctan\left(\frac{2}{3}\right)}_{\theta \text{ (in Q1)}}\right) = \cos\theta = \frac{3}{\sqrt{13}}$

$\theta = \arctan\left(\frac{2}{3}\right)$

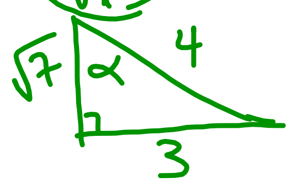
$\tan\theta = \frac{2}{3}$



b) $\tan\left(\underbrace{\sin^{-1}\left(\frac{3}{4}\right)}_{\alpha \text{ (in Q1)}}\right) = \tan\alpha = \frac{3}{\sqrt{7}}$

$\alpha = \sin^{-1}\left(\frac{3}{4}\right)$

$\sin\alpha = \frac{3}{4}$



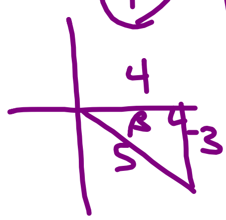
c) $\sec\left(\cos^{-1}\left(\frac{3x}{2}\right)\right) = \frac{1}{\cos\left(\cos^{-1}\left(\frac{3x}{2}\right)\right)} = \frac{1}{\frac{3x}{2}} = \frac{2}{3x}$

Ex 5: Evaluate these.

a) $\sec\left(\underbrace{\arctan\left(-\frac{3}{4}\right)}_{\beta \text{ in Q4}}\right) = \sec\beta = \frac{5}{4}$

$\beta = \arctan\left(\frac{-3}{4}\right)$

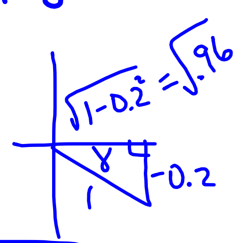
$\tan\beta = \frac{-3}{4}$



b) $\cot(\underbrace{\sin^{-1}(-0.2)}_{\gamma \text{ in Q4}}) = \cot\gamma$

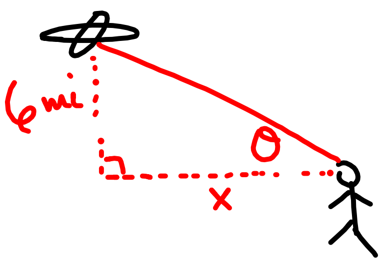
$\gamma = \sin^{-1}(-0.2)$

$\sin\gamma = -0.2$



$\Rightarrow \cot\gamma = \boxed{\frac{\sqrt{0.96}}{-0.2}}$

Ex 6: A plane flies at an altitude of 6 miles toward a point directly over an observer. Write the elevation angle θ as a function of x, the horizontal distance from the observer to a point on the ground directly below the airplane.



$\theta = ?$

$x > 0$

$\tan\theta = \frac{6}{x}$

$\Rightarrow \theta$ in Q1

$$\boxed{\theta = \tan^{-1}\left(\frac{6}{x}\right)}$$