3.5 Trigonometric Form of Complex Numbers

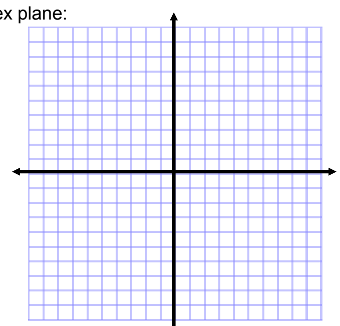
- Plot complex numbers in a complex plane.
- Determine the modulus and argument of complex numbers and write them in trigonometric form.
- Multiply and divide two complex numbers in trigonometric form.
- Use DeMoivre's Theorem to find powers of complex numbers.
- Determine the nth roots of complex numbers.
- What is the square root of i? Are there more than one of them?

Review : What is i ?

i i^2 i^3 i^4

Rectangular form of a complex number: $a + bi$

Complex plane:



$z_1 = 3+2i$

$z_2 = 1- 4i$

Absolute value of a complex number: $|a+bi| = \sqrt{a^2 + b^2}$

Add two complex numbers:

Multiply two complex numbers:

Trigonometric form of a complex number.

$z = a + bi$ becomes $z = r(\cos\theta + i\sin\theta)$

$r = |z|$ and the reference angle, θ' is given by $\tan\theta' = |b/a|$

Note that it is up to you to make sure θ is in the correct quadrant.

Example: Put these complex numbers in Trigonometric form.

$4 - 4i$

$-2 + 3i$

Writing a complex number in standard form:

Example: Write each of these numbers in a + bi form.

$\sqrt{2}(\cos 2\pi/3 + i \sin 2\pi/3)$

$20 (\cos 75^\circ + i \sin 75^\circ)$

Multiplying and dividing two complex numbers in trigonometric form:

$z_1 = 3(\cos 120^\circ + i \sin 120^\circ)$ $z_2 = 12 (\cos 45^\circ + i \sin 45^\circ)$

To multiply two complex numbers, you multiply the moduli and add the arguments.

$z_1 z_2 = r_1 r_2(\cos(ø_1+ø_2) + i \sin(ø_1+ø_2))$

To divide two complex numbers, you divide the moduli and subtract the arguments.

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(ø_1 - ø_2) + i \sin(ø_1 - ø_2))$

Please note that you must be sure your that in your answer
r is positive and $0 < \theta < 360^\circ$.

Here is an example. Find the product and quotient of these two complex numbers.

$z_1 = 3(\cos 150^\circ + i \sin 150^\circ)$ and $z_2 = 12 (\cos 275^\circ + i \sin 275^\circ)$

Powers of complex numbers

DeMoivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$z^n = r^n(\cos n\theta + i\sin n\theta)$

Example: Use DeMoivre's Theorem to find $(2-2i)^7$

Roots of complex numbers

Every number has two square roots.

The square roots of 16 are:

The square roots of 24 are:

The square roots of -81 are:

The square roots of -75 are:

Likewise, every number has three cube roots, four fourth roots, etc. (over the complex number system.)

So if we want to find the four fourth roots of 8 we solve this equation.

$x^4 = 16$

If we solve $x^6 - 1 = 0$ we can do some fancy factoring to get six roots.

Do you remember how to factor the sum/difference of two cubes?

Later we will solve this using a variation of DeMoivre's Theorem.

We can extend DeMoivre's Theorem for roots as well as powers.

$z = r(\cos\theta + i\sin\theta)$ has n distinct nth roots.

The first is $\sqrt[n]{r}\,(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$

and the others are found by adding 360°/n or $2\pi/n$ n-1 times to the angle of the first answer.

Thus for the previous two examples we write:

$x^4 = 8$

$x^6 - 1 = 0$

We will do this on the next page.

Two more:

Find the three cube roots of -8.

Find the five fifth roots of unity (1).

Now to solve the previous problem, $x^6 - 1 = 0$, we can use this theorem.

Start with $x^6 = 1$ We are looking for the six sixth roots of unity (1)

Finally we can answer the question: What are the two square roots of i ?

In summary ~ Powers and roots of a complex number in trigonometric form:

$z^n = r^n(\cos(n\phi) + i\sin(n\phi))$

$z^{1/n} = \sqrt[n]{r}\,(\cos(\phi/n) + i\sin(\phi/n))$
for the first root, with others 360º/n apart.

The cube of z (z to the third power):

The five fifth roots of z: