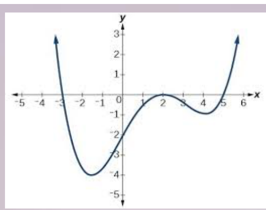

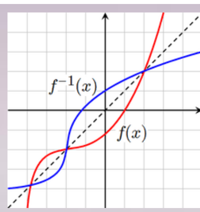

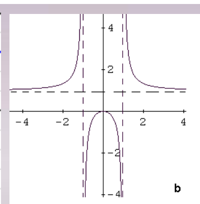

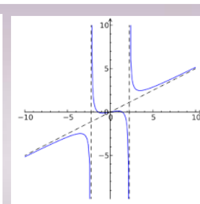

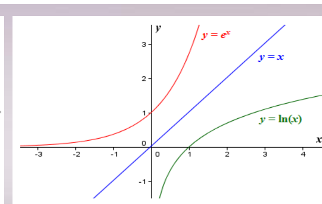

Math 1050 ~ College Algebra

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

30 Binomial Expansion

Learning Objectives

- Expand binomial powers using
  - Pascal’s Triangle
  - Binomial Theorem
- Find an indicated term in the expansion of a binomial.

Binomial Expansion binomial: two-termed expression

The square and cube of a binomial are used so frequently, we may want to memorize them.

Ex 1: $(a+b)^2 =$ $(a+b)(a+b)$

$= a^2 + 2ab + b^2$

$(♡ + ★)^2 = ♡^2 + 2♡★ + ★^2$

$(a+b)^3 =$ $(a+b)(a+b)(a+b)$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Given these, it is possible to determine the square or cube of more complicated binomials.

Ex 2: $(x\text{-}2y)^2 =$ $x^2 + 2(x)(-2y) + (-2y)^2$

[$♡ = x$ $★ = -2y$]

$= x^2 - 4xy + 4y^2$

$a = 3x, \ b = 2y$

$(3x+2y)^3 =$ $(3x)^3 + 3(3x)^2(2y) + 3(3x)(2y)^2 + (2y)^3$

$= 27x^3 + 54x^2y + 36xy^2 + 8y^3$

<u>Pascal's Triangle</u>

$(a+b)^0 = 1$

$(a+b)^1 = 1a+1b$

$(a+b)^2 = 1a^2+2ab+1b^2$

$(a+b)^3 = 1a^3+3a^2b+3ab^2+1b^3$

$(a+b)^4 = 1a^4 \ +4a^3b$

$+6a^2b^2 \ +4ab^3 \ +1b^4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Ex 3: Using the pattern above, expand these.

a) $(x\text{-}y)^5$ $=(x+(-y))^5$

$a=x$
$b=-y$

$=1x^5+5x^4(-y)+10x^3(-y)^2$
$+10x^2(-y)^3+5x(-y)^4+1(-y)^5$

$=\boxed{x^5-5x^4y+10x^3y^2 -10x^2y^3+5xy^4-y^5}$

b) $(2x\text{-}3y)^4$

$a=2x$
$b=-3y$

$=1(2x)^4+4(2x)^3(-3y)$
$+6(2x)^2(-3y)^2+4(2x)(-3y)^3$
$+1(-3y)^4$

$=\boxed{16x^4-96x^3y+216x^2y^2 -216xy^3+81y^4}$

The Binomial Theorem

To get to this theorem, we need to introduce a new notation.

Factorial

The factorial of a non-negative integer is defined as:

$0! = 1$ (by defn)

$n! = 1(2)(3)...(n)$

Ex 4: Evaluate the factorials of the integers 1 through 6.

$1! = 1$ $\quad$ $3! = 3 \cdot 2 \cdot 1 = 6$ $\quad$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$2! = 2 \cdot 1 = 2$ $\quad$ $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ $\quad$ $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Ex 5: Evaluate

a) $\frac{5!}{3!}$

$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$

$= \frac{5 \cdot 4 \cancel{3!}}{\cancel{3!}} = 5 \cdot 4$

$= 20$

b) $(8-5)!$

factorials do NOT distribute through addition

$3! = 6$

c) $(8-3)!$

$= 5!$

$= 120$

d) $\frac{17!}{14!}$

$= \frac{17 \cdot 16 \cdot 15 \cdot \cancel{14!}}{\cancel{14!}}$

$= 17(16)(15)$

$= 4{,}080$

e) $\frac{100!}{96!4!}$

$= \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot \cancel{96!}}{\cancel{96!}\,(4 \cdot 3 \cdot 2 \cdot 1)}$

$= 25(33)(49)(97)$

$= \boxed{3{,}921{,}225}$

<u>The Binomial Coefficient</u> $\binom{n}{k}$ n, k are nonnegative integers

$n \geq k$

read as "n choose k"

also notate as ${}_nC_k$ or $\binom{n}{k}$

defn

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

Ex 6: Evaluate these binomial coefficients.

a) $\binom{5}{0}$ b) $\binom{10}{6}$ c) $\binom{8}{3}$ d) $\binom{8}{5}$ e) $\binom{8}{8}$

a) $= \frac{5!}{0!(5-0)!} = \frac{\cancel{5!}}{\cancel{5!}} = 1$

b) $= \frac{10!}{6!\,4!} = \frac{10\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6!}}{\cancel{6!}\,(\cancel{4}\cdot 3\cdot \cancel{2})} = 10\cdot 3\cdot 7 = 210$

c) $= \frac{8!}{3!\,5!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{(\cancel{3\cdot 2})\cancel{5!}} = 8\cdot 7 = 56$

d) $= \frac{8!}{5!(3!)} = 56$

e) $= \frac{\cancel{8!}}{\cancel{8!}\,0!} = 1$

Ex 7: Show that $(a+b)^4 = \sum_{k=0}^{4}\binom{4}{k}a^{4-k}b^k$ $a, b \neq 0$

$$\sum_{k=0}^{4}\binom{4}{k}a^{4-k}b^k = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

$$= \frac{\cancel{4!}}{0!\cancel{4!}}a^4 + \frac{4!}{1!3!}a^3b + \frac{4!}{2!2!}a^2b^2 + \frac{4!}{3!1!}ab^3 + \frac{\cancel{4!}}{\cancel{4!}0!}b^4$$

$$= a^4 + 4a^3b + \frac{\cancel{4}\cdot 3\cdot 2}{\cancel{2\cdot 2}}a^2b^2 + 4ab^3 + b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$= (a+b)^4$$

<u>The Binomial Theorem</u> $\boxed{(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^k}$

<u>NOTE</u>: The $\binom{n}{k}$ are exactly the numbers in Pascal's Δ.

Ex 8: Use this theorem to expand $(3x\text{-}2)^5$. $a=3x$, $b=-2$

$$(3x-2)^5 = \binom{5}{0}(3x)^5 + \binom{5}{1}(3x)^4(-2)^1 + \binom{5}{2}(3x)^3(-2)^2$$

$$+\binom{5}{3}(3x)^2(-2)^3 + \binom{5}{4}(3x)(-2)^4 + \binom{5}{5}(-2)^5$$

$$= 1(3^5x^5) + 5(3^4x^4)(-2) + 10(3^3x^3)(4) + 10(3^2x^2)(8)$$

$$+ 5(3x)(16) + 1(-32)$$

$$= \boxed{243x^5 - 810x^4 + 1080x^3 + 720x^2 + 240x - 32}$$

Ex 9: Consider $(2x\text{-}y)^8$. Find the term that contains x^3 and the term that contains y^4. $a=2x$, $b=-y$

1) x^3 term: $\binom{8}{3}(2x)^3(-y)^5 = \frac{8!}{3!\,5!}(8x^3)(-y^5)$

$$= \frac{8\cdot7\cdot\cancel{6}\cdot\cancel{5!}}{(\cancel{3\cdot2})\cancel{5!}}(-8x^3y^5)$$

$$= -448x^3y^5$$

2) y^4 term: $\binom{8}{4}(2x)^4(-y)^4$

$$= \frac{8!}{4!\,4!}(2^4x^4y^4) = \frac{\cancel{8}\cdot7\cdot6\cdot5\cdot\cancel{4!}}{(\cancel{4}\cdot3\cdot\cancel{2})\cancel{4!}}(16x^4y^4)$$

$$= 1120x^4y^4$$