8.3 The inverse of a square matrix

You will:

- Verify that matrices are inverses of each other.
- Determine the inverse of a 2x2 matrix if it exists.
- Use Gauss-Jordan elimination to determine the inverse of a 3x3 matrix.
- Use inverse matrices to solve systems of linear equations.

Let A be an $n \times n$ matrix and I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$A A^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the inverse of A. The symbol A^{-1} is read *A inverse.*

A^{-1} "A inverse"

Example 1:

Which two are inverses?

$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+-1 \\ -1+-2 & -2+2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$\Rightarrow B \neq A^{-1}$

$$AC = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+-1 & 1+-1 \\ -2+2 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow C = A^{-1}$

and $A = C^{-1}$

Process for finding A^{-1} :

Augment A with I
Perform row operations until the left side of the augmented matrix looks like I.
The right side is A^{-1}

Example 2:

a) Find the inverse if it exists:

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

$$\overbrace{\begin{matrix} -3 & -6 \end{matrix}}^{A} \quad \overbrace{\begin{matrix} -3 & 0 \end{matrix}}^{I}$$

$$(-3)\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] \Rightarrow (-2)\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

b) Find the inverse if it exists:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix} \quad (-2)(-3)\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$(-1)\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right] \quad (-4)\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 1 & 5 & 1 & -1 & 1 \end{array}\right]$$

$$\left(\tfrac{-1}{20}\right)\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -20 & -7 & 5 & -4 \\ 0 & 1 & 5 & 1 & -1 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{-1}{4} & \frac{1}{5} \\ 0 & 1 & 5 & 1 & -1 & 1 \end{array}\right]$$

$$(-5)\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{-1}{4} & \frac{1}{5} \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{-1}{4} & \frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3/4 & 1/4 & 0 \\ 7/20 & -1/4 & 1/5 \end{bmatrix}$$

Using Matrix Algebra to solve systems of linear equations.

Example 3:

a) Find A^{-1}:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

$$(-3)\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$(-2)\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right] (-3)\ (-2)$$

b) Use the inverse above to solve this system of equations:

x + 2y + 2z = 0
3x +7y + 9z = 1
-x - 4y - 7z = 2

$$(-2)\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 11 & -4 & -2 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{B}$$

system of linear eqns

$AX = B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

$$A^{-1}B = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$(3\times3)(3\times1) \rightarrow (3\times1)$

$$= \begin{bmatrix} 0+6+8 \\ 0+-5+-6 \\ 0+2+2 \end{bmatrix} = \begin{bmatrix} 14 \\ -11 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -11 \\ 4 \end{bmatrix} \iff \begin{array}{l} x = 14 \\ y = -11 \\ z = 4 \end{array}$$

$(14, -11, 4)$