MATH 1010 ~ Intermediate Algebra

Chapter 7: RADICALS AND COMPLEX NUMBERS

Section 7.6: Complex Numbers

Objectives:

- Write the square roots of negative numbers in i-form and perform operations on them.
- Determine the equality of two complex numbers
- Add, subtract and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.

$$\sqrt{-1} = i$$

$$\sqrt{-45}$$

$$3-2i$$

The Imaginary Unit i

$$\text{defn}\quad \sqrt{-1} = i$$

Powers of i

$$i^1 = i$$

$$i^2 = \sqrt{-1}\sqrt{-1} = -1$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

$$i^6 = i^4 \cdot i^2 = -1$$

$$i^7 = i^4 \cdot i^3 = -i$$

i^{th}	i^{th}
1	i
2	-1
3	$-i$
4	1
5	i
6	-1
7	$-i$
8	1

① Simplify these.

a) $\sqrt{-9} = \sqrt{-1}\sqrt{9} = i(3) = \boxed{3i}$

b) $\sqrt{-27} = \sqrt{-1}\sqrt{9}\sqrt{3} = i(3)\sqrt{3} = 3i\sqrt{3} \sim 3\sqrt{3}\,i$

c) $\sqrt{-\frac{81}{8}} = \frac{\sqrt{-1}\sqrt{81}}{\sqrt{4}\sqrt{2}} = \boxed{\frac{9i}{2\sqrt{2}}}$

Standard Form of Complex Numbers

$$a + bi$$

real part

imaginary part

$a, b \in \mathbb{R}$

$i = \sqrt{-1}$

Equality of Two Complex Numbers

$$a + bi = c + di$$

iff $a = c$ and $b = d$

(iff = if and only if)

Operations on Complex Numbers

Addition and Subtraction:

$(a \pm bi) + (c \pm di) = (a \pm c) + (b \pm d)i$

② EXAMPLE

Combine and simplify these.

$a)\quad (3-i)+(-2+5i)$

$= (3+-2) + (5-1)i = 1+4i$

$b)\quad 2-i+\sqrt{25}-\sqrt{-49}$

$= 2-i+5-\sqrt{-1}\sqrt{49}$

$= 2-i+5-i(7)$ → $= 7-8i$ $(a=7,\ b=-8)$

$c)\quad 7+3i+1-\sqrt{-8}-\sqrt{-4}$

$= 7+3i+1-\sqrt{-1}\sqrt{4}\sqrt{2}-\sqrt{-1}\sqrt{4}$

$= 8+3i-2\sqrt{2}i-2i$

$= 8+(1-2\sqrt{2})i$

$(a=8,\ b=1-2\sqrt{2})$

Multiplying Complex Numbers

$(a+bi)(c+di)$

$= ac + adi + bci + bdi^2$ $\quad (i^2 = -1)$

$= ac + adi + bci - bd$

$= (ac - bd) + (ad + bc)i$

③ EXAMPLE

Multiply and simplify these.

$\sqrt{-4} = \sqrt{-1}\sqrt{4} = i(2)$

a) $(2-3i)(\sqrt{-4}) = (2-3i)2i$

$= 4i - 6i^2 = 4i - 6(-1) = \boxed{6+4i}$

b) $(3-4i)(2+5i) = 6 + 15i - 8i - 20i^2$

$= 6 + 7i + 20 = \boxed{26 + 7i}$

c) $(4-i)(4+i)$

$= 16 + 4i - 4i - i^2$

$= 16 - (-1) = 16 + 1 = \boxed{17}$

Complex Conjugates

$a+bi$ conjugate $a-bi$ $(i^2=-1)$

$(a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2$

$= a^2 - b^2(-1) = a^2 + b^2$

Complex # multiplied by its conjugate gives $\mathbb{R}$ #

④ EXAMPLE

Determine the conjugate of each of these and multiply the number and the conjugate.

$a)\quad 7-3i$ conjugate $7+3i$

$(7-3i)(7+3i) = 49 + 21i - 21i - 9i^2 = 49+9$

$= \boxed{58}$

$b)\quad -8+2i$ conjugate $-8-2i$

$(-8+2i)(-8-2i) = 64 + 16i - 16i - 4i^2$

$= 64 + 4 = \boxed{68}$

$c)\quad 9\sqrt{3} - 2\sqrt{5}i$ conjugate $9\sqrt{3} + 2\sqrt{5}i$

$(9\sqrt{3} - 2\sqrt{5}i)(9\sqrt{3} + 2\sqrt{5}i)$

$= 81(3) + 18\sqrt{15}i - 18\sqrt{15}i - 4(5)i^2$

$= 243 - 20(-1)$

$= 243 + 20 = \boxed{263}$

Division of Complex Numbers

$$\left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) = \frac{ac-adi+bci-bdi^2}{c^2+d^2}$$

(+bd)

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i$$

⑤ EXAMPLE

Determine the quotient of these. (answer in standard form)

a) $\left(\frac{2-3i}{2i}\right)\left(\frac{i}{i}\right) = \frac{2i-3i^2}{2i^2} = \frac{2i+3}{-2}$

$= -1i + \frac{-3}{2}$ or $\boxed{\frac{-3}{2} - i}$

b) $\left(\frac{6}{4+i}\right)\left(\frac{4-i}{4-i}\right) = \frac{24-6i}{16-i^2} = \frac{24-6i}{17} = \boxed{\frac{24}{17} - \frac{6}{17}i}$

c) $\left(\frac{2-4i}{1+3i}\right)\left(\frac{1-3i}{1-3i}\right) = \frac{2-6i-4i+12i^2}{1-9i^2}$

$= \frac{2-10i-12}{1+9}$

$= \frac{-10-10i}{10} = \frac{-10}{10} - \frac{10}{10}i = \boxed{-1-i}$

A few more things:

n	i^n
1	i
2	-1
3	$-i$
4	1
5	i
6	-1
7	$-i$
8	1

a) $i^{25} = i^{24} i^1 = (i^4)^6 i^1$

$= 1^6 i = i$

$4\overline{)25}$ 6, -24, 1

b) $i^{177} = i^{176} i^1$

$= i$

$4\overline{)177}$ 44 R 1, -16, 17, -16, 1

c) $i^{104} = (i^4)^{26}$

$= 1^{26} = 1$

$4\overline{)104}$ 26, -8, 24, -24, 0

Remember this:

$$\sqrt{-72}\sqrt{-8} =$$

$= \sqrt{-1}\sqrt{72}\sqrt{-1}\sqrt{8}$

$= i^2\sqrt{9 \cdot 8 \cdot 8}$

$= -1(3)(8) = -24$

WARNING:

$\sqrt{-72}\sqrt{-8} \neq \sqrt{72 \cdot 8}$

$$\frac{\sqrt{-72}}{\sqrt{-8}} = \frac{\sqrt{-1}\sqrt{72}}{\sqrt{-1}\sqrt{8}}$$

$= \frac{i 3\sqrt{8}}{i\sqrt{8}}$

$= \frac{3i}{i}\left(\frac{i}{i}\right)$

$= \frac{3(-1)}{(-1)} = 3$

$\frac{3i}{i} = 3$