2023 State Math Competition

Junior Exam

Version A

Instructions:

- Make sure to write your name and mark the version on your answer sheet. Write your school name in the ID space and your grade in the Section space.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- No materials (textbooks, notes, calculator, internet, etc) allowed.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. The wholesale price of an item was increased by 25% when the item arrived in the store. After a few weeks, the item was put on sale for the wholesale price. By what percent was the price reduced for the sale?

   (a) 15
   (b) 20
   (c) 16
   (d) 36
   (e) 25

   Correct answers: (b)

   Explanation: We have $P_1 = 1.25P_0 = \frac{5}{4}P_0$. Thus, $P_0 = \frac{4}{5}P_1 = (1 - 0.2)P_1$.

2. Every angle of a certain triangle is acute and the size of the largest angle is 5 times that of the smallest angle. If all of the angles have integer degree measurements, then the smallest angle is

   (a) 11°
   (b) 17°
   (c) 14°
   (d) 18°
   (e) 12°

   Correct answers: (b)

   Explanation: Denote the size of the angles by x, y, z such that $z < y < x$. Then $z < y < x < 90^\circ$ and $x = 5z$. Then $5z < 90^\circ$, which gives us that $z < 18^\circ$. But we know that $x + y + z = 180^\circ$, but since $x + y + z < x + x + z = 11z$, and so $z > \frac{180^\circ}{11} > 16^\circ$. Therefore $16^\circ < z < 18^\circ$, so we must have that $z = 17^\circ$.

3. The number 2532645918 is divisible by

   (a) both 3 and 11 but not 4
   (b) neither 3 nor 11 nor 4
   (c) both 3 and 4 but not 11
   (d) 3 but neither 11 nor 4
   (e) both 4 and 11 but not 3

   Correct answers: (a)

Explanation: You can just do the division or you can use divisibility tricks:
Checking divisibility by 3: 2+5+3+2+6+4+5+9+1+8 = 45 is divisible by 3 $\Rightarrow 3 \mid 2532645918$.
Checking divisibility by 11: (2+3+6+5+1) - (5+2+4+9+8) = -11 is divisible by 11 $\Rightarrow 11 \mid 2532645918$.
Checking divisibility by 4: Last 2 digits of 2532645918 is 18 which is not divisible by 4 $\Rightarrow 4 \nmid 2532645918$.
Thus 2532645918 is divisible by both 3 and 11 but not 4.

4. How many pairs (b, c) exist such that $4x^2 + bx + c$ has a repeated real root, when b and c are each allowed to be positive integers less than or equal to 100?

   (a) 36
   (b) 4
   (c) 100
   (d) 50
   (e) 10

Correct answers: (e)

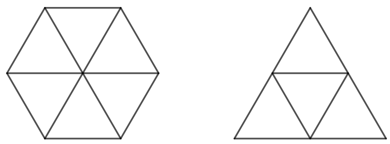
Explanation: The equation is $4x^2 + bx + c = 0$ will have repeated real roots if the discriminant $b^2 - 4(4)c = 0 \Rightarrow b^2 = 16c$. This implies that c must be a perfect square and we have the possible values $c = 1, 4, 9, 16, 25, 36, 49, 64, 81, 100$. So there are 10 possible choices (b, c).

5. A regular hexagon and an equilateral triangle have equal circumferences. If you divide the area of the hexagon by the area of the triangle, what number do you get?

   (a) 2/1
   (b) 4/3
   (c) 8/3
   (d) 4/1
   (e) 3/2

Correct answers: (e)

Explanation: The circumferences being equal, the side of triangle is twice as long as the side of the hexagon. Let a be the side length of the hexagon. Then the hexagon can be divided into 6 equilateral triangles of side length a, whereas the triangle can be split into 4 such triangles (see image below). Therefore the ratio of their areas is $6/4$, i.e. $3/2$.



6. How many integers x exist such that $\dfrac{x^3+3x^2-x-10}{x+2}$ is an integer?

   (a) 2
   (b) 6
   (c) 4
   (d) 0
   (e) 8

   Correct answers: (b)

   Explanation: We do polynomial long division to see that

   $$\frac{x^3+3x^2-x-10}{x+2} = x^2+x-3-\frac{4}{x+2}$$

   So this will only an integer if $x+2$ divides -4. Since -4 has 6 divisors $(\pm 4, \pm 2, \pm 1)$, we conclude that there are 6 possible values of x. (The values are $-6, -4, -3, -1, 0, 2$.)

7. Let $a \vee b$ represent the operation on two numbers, a and b, which selects the larger of the two numbers, with $a \vee a = a$. Let $a \wedge b$ represent the operation which selects the smaller of the two numbers, with $a \wedge a = a$. Which of the following three rules are/is valid for all a, b, c?

   (1) $a \vee b = b \vee a$,

   (2) $a \vee (b \vee c) = (a \vee b) \vee c$,

   (3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

   (a) (1) and (2) only
   (b) (1) and (3) only
   (c) all three
   (d) (1) only
   (e) (2) only

   Correct answers: (c)

   Explanation: The correctness of rule (1) is obvious. To establish rule (2) observe that the left side selects the largest of all three numbers by comparing a with the larger of b and c, while the right side selects the largest of all three by comparing c with the larger of a and b.

   To establish rule (3), suppose first that a is the smallest of the three numbers. Then the left side selects it. Each parenthesis on the right selects it and the larger of two equal numbers a is a. If b or c is smallest, say b, then the left side selects the smaller of a and c, and so does the right. If c is the smallest, we argue the same way.

8. There are 24 coins in a bag, each of which is a dime or a quarter. There are an even number of dimes in the bag. If half of the dimes were removed and replaced with the same number of quarters, the value in the bag would increase by \$1.05. How much are the coins in the bag worth?

   (a) \$3.70
   (b) \$3.90
   (c) \$4.20
   (d) \$4.60
   (e) \$3.30

   Correct answers: (b)

   Explanation: Replacing 1 dime with 1 quarter increases the value by \$0.15. Replacing half the dimes with quarters would increase the value by \$1.05, so $\frac{\$1.05}{\$0.15} = 7$ coins get replaced. That means there are 14 dimes in the bag, and there must be 10 quarters. The total value is $\$0.10 \cdot 14 + \$0.25 \cdot 10 = \$3.90$.

9. What is the units digit of 3^{2023}?

   (a) 3
   (b) 4
   (c) 1
   (d) 9
   (e) 7

   Correct answers: (e)

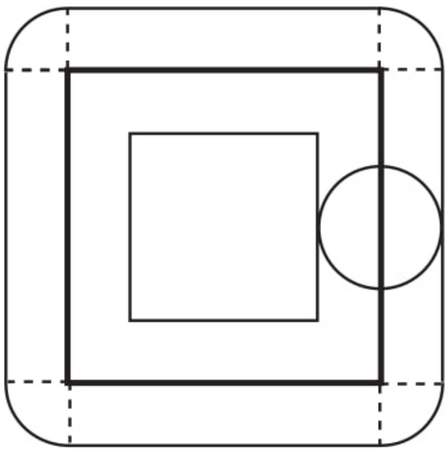
   Explanation: The possible numbers on the unit place is $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$ and this pattern repeats. To find the last digit of 3^{2023}, take 2023 mod 4 which is 3. That is, 2023 is 3 more than a multiple of 4. So the number in units place is the same digit in units place of 3^3 which is 7.

10. Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers and rounded to the nearest whole number?

   (a) 40
   (b) 42
   (c) 24
   (d) 39
   (e) 27

   Correct answers: (d)

   Explanation: At any point on Charlyn's walk, she can see all the points inside a circle of radius 1 km. The portion of the viewable region inside the square consists of the interior of the square except for a smaller square with side length 3 km. This portion of the viewable region has area $(25 - 9)km^2$. The portion of the viewable region outside the square consists of four rectangles, each 5 km by 1 km, and four quarter-circles, each with a radius of 1 km. This portion of the viewable region has area $4(5 + \frac{\pi}{4}) = (20 + \pi)km^2$. The area of the entire viewable region is $36 + \pi \approx 39km^2$.



11. Define $n! = 1 \cdot 2 \cdots n$ for all integers $n \geq 1$. What is $\sum_{k=1}^{100} k \cdot k!$.

   (a) $101! - 1$
   (b) $101! + 100!$
   (c) $101! + 101$
   (d) $101!$
   (e) $101! - 101$

   Correct answers: (a)

Explanation: Note that $k \cdot k! = (k+1-1) \cdot k! = (k+1) \cdot k! - k! = (k+1)! - k!$.
Summing from 1 to 100 cancels most terms and we are left with $101! - 1!$.

12. The Jones family is going on a summer vacation. They will depart on the last day of the month. If you multiply the following four numbers:

    - the day of their departure
    - the month of their return
    - the number of pets they own
    - the length of their vacation,

    you get $14384 = 2^4 \cdot 31 \cdot 29$. How many pets does the Jones family own?

    (a) 3
    (b) 5
    (c) 2
    (d) 4
    (e) 1

    Correct answers: (c)

    Explanation: The only divisor that could also be the last day of a month is 31 (29 is also a possibility — February in a leap year — but then it wouldn't be a summer vacation). There are multiple divisors that could be the month of their return: $1, 2, 4, 8$. Again, only 8 (August) corresponds to a summer vacation. This leaves $2 \cdot 29$, so we conclude that their vacation will last 29 days, and that they have 2 pets. (Technically, they could have 29 pets and go on a 2-day vacation... but that's not realistic. Also, 29 is not among the given choices).

13. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

    (a) $\frac{1}{2}$
    (b) $\frac{2}{5}$
    (c) $\frac{3}{10}$
    (d) $\frac{7}{10}$
    (e) $\frac{3}{5}$

    Correct answers: (e)

    Explanation: Imagine drawing until only one chip remains. If the remaining chip is red, then that draw would have ended when the second white chip was removed. The last chip will be red with probability $\frac{3}{5}$. You could also enumerate all 10 possibilities (e.g. $rwwrr$...) and determine the answer that way.

14. The polynomial $f(x) = (2+x)^n$ can be expanded as $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ where n is a positive integer. If $\sum_{j=0}^{n} c_j = 81$, then the largest coefficient c_j of f is

    (a) 24
    (b) 32
    (c) 64
    (d) 16
    (e) 36

    Correct answers: (b)

Explanation: We can write $81 = \sum_{j=0}^{n} c_j = f(1) = 3^n \Rightarrow n = 4$.
The Binomial expansion of $(2+1)^4$, shows that the coefficients are $\binom{4}{0}2^0 = 1$, $\binom{4}{1}2^1 = 8$, $\binom{4}{2}2^2 = 24$, $\binom{4}{3}2^3 = 32$ and $\binom{4}{4}2^4 = 16$. Thus, the largest coefficient c_j is 32.

15. Let $a = 3^{13} \cdot 5^7 \cdot 7^{20}$. How many positive integers k have the property that k divides a and 105 divides k?

(a) 1824
(b) 1820
(c) 1771
(d) 2036
(e) 1368

Correct answers: (b)

Explanation: Since k divides a, it must have a factorization $3^x \cdot 5^y \cdot 7^z$, for some integers $0 \leq x \leq 13$, $0 \leq y \leq 7$, and $0 \leq z \leq 20$. Since $105 = 3 \cdot 5 \cdot 7$ divides k, the factorization is restricted so that $1 \leq x \leq 13$, $1 \leq y \leq 7$, and $1 \leq z \leq 20$. Therefore, there are $13 \cdot 7 \cdot 20 = 1820$ different factorizations, and hence values, of k.

16. Eight players are participating in a chess tournament. They each play one another once, getting 1 point for a win, $\frac{1}{2}$ for a draw, and 0 for a loss. This is their ranking at the end of the tournament:

1. Amelia
2. Bill
3. Clyde
4. Diane
5. Ellie
6. Faye
7. Gertrude
8. Hayden

At the end of the tournament, no two participants have the same number of points. Bill's score is equal to the sum the last four players' scores. How did the game between Diane and Ellie go?

(a) we can rule out Ellie's victory, but both Diane's win and a draw are possible
(b) not enough information to determine
(c) It was a draw
(d) Ellie won
(e) Diane won

Correct answers: (e)

Explanation: The sum of the last four players' scores is at least 6, because they played six games among themselves. On the other hand, Bill has at least 6 points. Indeed, the maximum number of points once can win is 7, but Bill cannot have 7 because then he would have been first. Furthermore, he cannot have 6.5 points because that would also mean that nobody is better than him. Thus, Bill has exactly six points, and the last four players' scores add up to 6. But that means that the last four players have not won any games against the first four. In other words, Diane won her game against Ellie.

17. The number of solutions to $2^{2x} - 3^{2y} = 55$, in which x and y are integers, is:

(a) more than three, but finite
(b) three
(c) zero
(d) two
(e) one

Correct answers: (e)

Explanation:

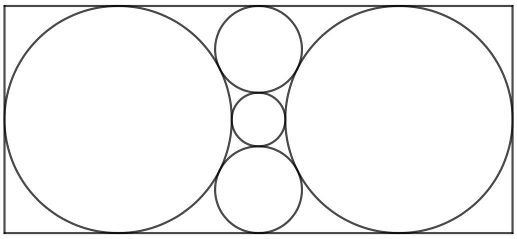
$$2^{2x} - 3^{2y} = (2^x + 3^y)(2^x - 3^y) = 11 \cdot 5 = 55 \cdot 1.$$

There are two possible solutions,

$$\begin{cases} 2^x + 3^y = 11 \\ 2^x - 3^y = 5 \end{cases} \quad \text{or} \quad \begin{cases} 2^x + 3^y = 55 \\ 2^x - 3^y = 1 \end{cases}.$$

Using the elimination method to solve these two systems of equations, we can see that the first system equations has unique integral solution $(x, y) = (3, 1)$, while the second system of equations has no integral solutions.

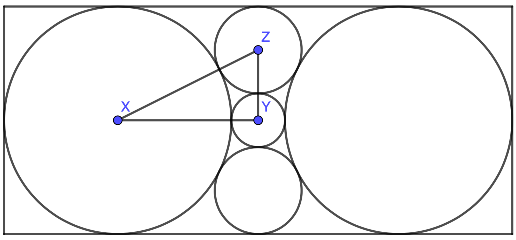
18. The shorter side of the rectangle pictured below has length 1. How long is the longer side?



(a) $\sqrt{3}$
(b) 2
(c) $\sqrt{6}$
(d) $\sqrt{5}$
(e) $\sqrt{2}$

Correct answers: (d)

Explanation: Consider the right triangle (pictured); let y be the radius of the smallest, and let z be the radius of the mid-sized circle. The radius of the largest circle is $1/2$.

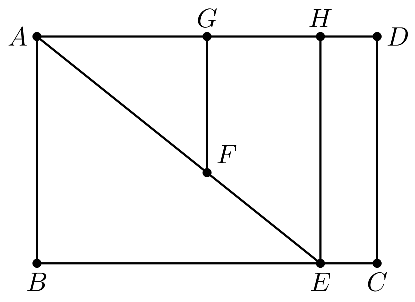


Applying the Pythagorean theorem, we get

$$(\frac{1}{2} + z)^2 = (y + z)^2 + (\frac{1}{2} + y)^2.$$

Furthermore, we see that $4z + 2y = 1$. Solving this for y and plugging into the quadratic expression above, we get a quadratic equation $4y^2 + 8y - 1 = 0$. Its positive solution is $y = \frac{\sqrt{5}}{2} - 1$. The length of the longer side of the rectangle is $2 + 2y = \sqrt{5}$.

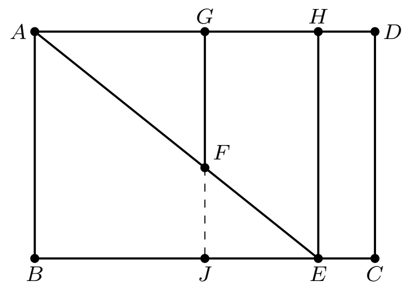
19. In rectangle $ABCD$, point E is on $\overline{BC}$ so that $AE = 7$, and point F is on $\overline{AE}$ so that $AF = 4$. G and H are on $\overline{AD}$ so that $\overline{FG} \parallel \overline{EH} \parallel \overline{AB}$, and $GH = 2$. What is the length AB?



(a) 5
(b) $\frac{9}{2}$
(c) $\frac{7\sqrt{5}}{3}$
(d) $3\sqrt{3}$
(e) $\frac{10\sqrt{2}}{3}$

Correct answers: (c)

Explanation: One method is to extend $\overline{GF}$ to reach $\overline{BE}$ at a point J. Then $JE = GH = 2$. Also $FE = AE - AF = 3$. By the Pythagorean theorem, $\overline{JE}^2 + \overline{FJ}^2 = \overline{FE}^2$, so $2^2 + \overline{FJ}^2 = 3^2$, and $FJ = \sqrt{5}$. $\triangle ABE$ is similar to $\triangle FJE$. Therefore, $\frac{AB}{\sqrt{5}} = \frac{7}{3}$. So $AB = \frac{7\sqrt{5}}{3}$.



20. Consider 8-step paths that start at $(-2, -2)$ and end at $(2, 2)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths are there that do not include $(0, 0)$?

(a) 48
(b) 34
(c) 64
(d) 24
(e) 28

Correct answers: (b)

Explanation: First, we can count all of the 8-step paths. To go from $(-2, -2)$ to $(2, 2)$, we must take a right-step 4 times and an up-step 4 times. By ordering these steps, we get all the possible 8-step paths. Use an R to indicate a right-step and a U to indicate an up-step. We want to count the different ways to order RRRRUUUU. There are $\binom{8}{4} = \frac{8!}{4!4!} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1} = 70$ paths.
Next, we can subtract the number of 8-step paths that go through $(0, 0)$. These paths start by going from $(-2, -2)$ to $(0, 0)$. There are $\binom{4}{2} = 6$ ways to do that. Then the second half of the path goes from $(0, 0)$ to $(2, 2)$. There are

$\binom{4}{2} = 6$ ways to do that. Overall there are $6 \cdot 6 = 36$ paths from $(-2, -2)$ to $(2, 2)$ that include $(0, 0)$. Therefore, $70 - 36 = 34$ paths do not go through $(0, 0)$.