State Junior Mathematics Contest
Spring 2008

1. If $a + b = 4$ and $a^2 + b^2 = 9$, then $a^3 + b^3 =$

(a) 22 (b) 16 (c) 64 (d) 28 (e) 36

$a = 4 - b$

$(4-b)^2 + b^2 = 9$

$16 - 8b + 2b^2 = 9$

$2b^2 - 8b + 7 = 0$

$b = \dfrac{8 \pm \sqrt{64 - 4(2)(7)}}{4}$

$b = \dfrac{4 \pm \sqrt{2}}{2} \Rightarrow a = 4 - \left(2 \pm \frac{\sqrt{2}}{2}\right) = 2 \mp \frac{\sqrt{2}}{2}$

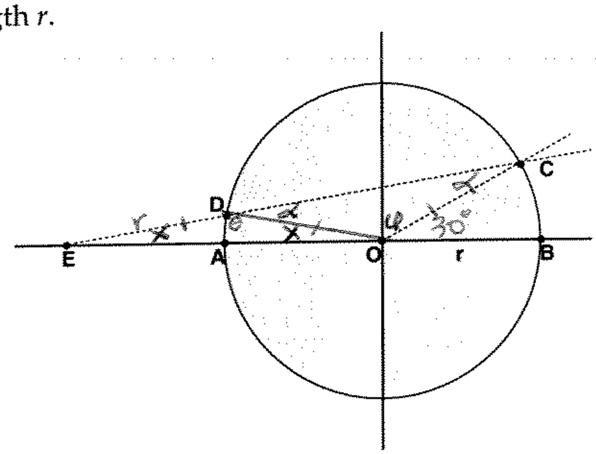
$a^3 + b^3 = \left(2 - \frac{\sqrt{2}}{2}\right)^3 + \left(2 + \frac{\sqrt{2}}{2}\right)^3$

$= 2^3 + 3(2^2)\left(-\frac{\sqrt{2}}{2}\right) + 3(2)\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^3$

$+ 2^3 + 3(2^2)\left(\frac{\sqrt{2}}{2}\right) + 3(2)\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^3$

$= 8 + 8 + 3 + 3 = 22$

2. Points A, B, C lie on a circle with radius r centered at O. Segment DE has length r.



If $m(< BOC) = 30°$, then $M(< BEC) =$

(*a*) 15°
(*b*) 12°
(*c*) 10°
(*d*) 20°
(*e*) cannot be determined without more infomation

$x = ?$

ΔEDO is isosceles $\Rightarrow$ base angles x are same

$2x + \theta = 180°$ but also $\theta + \alpha = 180°$

$\Rightarrow \alpha = 2x$

ΔDOC is isosceles $\Rightarrow \angle ODC \cong \angle OCD$

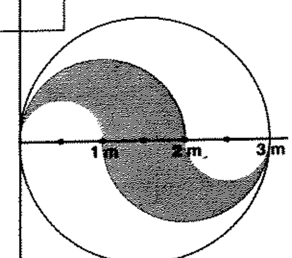
$\varphi + 2\alpha = 180°$ and $x + \varphi + 30° = 180°$

$\varphi = 180° - 2\alpha \quad \Rightarrow \quad x + 180° - 4x + 30° = 180°$

$\varphi = 180° - 4x \qquad -3x = -30°$

$x = 10°$

3. The circle to the right has diameter $3\,m$. Find the area of the shaded region if its boundary consists of semicircles:



(*a*) $3\pi\, m^2$ (*b*) $\pi\, m^2$ (*c*) $\frac{3}{2}\pi\, m^2$ (*d*) $\frac{3}{4}\pi\, m^2$
(*e*) $\frac{3}{8}\pi\, m^2$

$$A_{shaded} = 2\left(\frac{\pi(1^2)}{2} - \frac{\pi\left(\frac{1}{2}\right)^2}{2}\right)$$

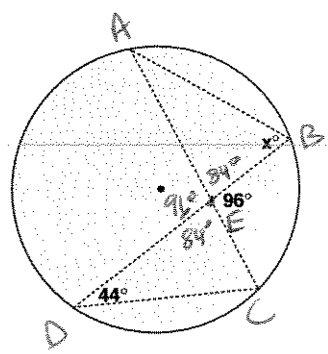
$$= \pi - \frac{1}{4}\pi = \frac{3}{4}\pi \quad m^2$$

4. If f is a function such that $f(x-1) = x^2 - 3x + 5$ then $f(x+1) = ?$

(a) $x^2 + x + 3$
(b) $x^2 - x + 3$
(c) $x^2 + x$
(d) $x^2 - 3x + 7$
(e) none of these

We know $f(x-1) = x^2 - 3x + 5$
let $y = x-1 \Rightarrow x = y+1$
Then
$f(y) = f(x-1) = x^2 - 3x + 5 = (y+1)^2 - 3(y+1) + 5$
$f(y) = y^2 + 2y + 1 - 3y - 3 + 5$
$f(y) = y^2 - y + 3$
or $f(x) = x^2 - x + 3$
$\Rightarrow f(x+1) = (x+1)^2 - (x+1) + 3$
$= x^2 + 2x + 1 - x - 1 + 3$
$= x^2 + x + 3$

5. Chords are drawn in a circle as shown. The value of x is



(a) 44 (b) 48 (c) 52 (d) 34 (e) 84

$\angle AED \cong \angle BEC$ vertical angles
$\Rightarrow m\angle AEB = m\angle CED = 84^\circ$ (supplementary angles)

$m\angle ACD = 180^\circ - 44^\circ - 84^\circ$
$= 52^\circ$

and $\angle ACD$ and $\angle ABD$ cut the same arc
$\Rightarrow \angle ACD \cong \angle ABD$
$\Rightarrow m\angle ABD = x = 52^\circ$

6. A 3×3 magic square uses integers $1, 2, \ldots, 9$ once each in such a way that each column, each row, and each diagonal sums to 15. Find the value of N for the magic square, a portion of which is shown below:

8		
N		7

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

8	$7-2N$	$2N$
N	$8-N$	7
$7-N$	$3N$	$8-2N$

(3) $7-N+8-N+X=15 \Rightarrow X=2N$

(1) $\Rightarrow$ middle $=15-N-7=8-N$

(5) top-middle $=15-8-2N=7-2N$

(2) bottom $=15-8-N$ $=7-N$

(4) $15-2N-7$ $=8-2N$

(6) bottom-middle $=15-(7-2N)-(8-N)$ $=3N$

3N can only be as big as $9 \Rightarrow N \leq 3$

if $N=1$, then $8-N=7$, but 7 is taken.

if $N=2$, then $8-N=3N$.

$\Rightarrow N=3$

8	1	6
3	5	7
4	9	2

7. If $i=\sqrt{-1}$, then $\dfrac{8-4i}{4+2i}=$

(*a*) $2-\frac{8}{3}i$ (*b*) 2 (*c*) $2-2i$ (*d*) $\frac{6}{5}-\frac{8}{5}i$ (*e*) $\frac{10}{3}-\frac{8}{3}i$

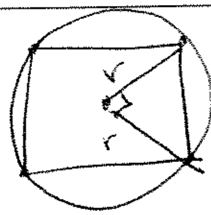
$$\left(\frac{8-4i}{4+2i}\right)\left(\frac{4-2i}{4-2i}\right)=\frac{32-16i-16i+8i^2}{16-4i^2}$$

$$=\frac{32-32i-8}{16+4}=\frac{24-32i}{20}$$

$$=\frac{24}{20}-\frac{32}{20}i=\frac{6}{5}-\frac{8}{5}i$$

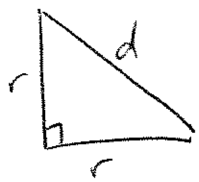
8. The ratio of the circumference of a circle to the perimeter of an inscribed square is:

(a) $\frac{\pi\sqrt{2}}{3}$ (b) $\frac{\pi\sqrt{2}}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$ (e) none of these



$$\frac{C_o}{P_\square} = \frac{2\pi r}{4\sqrt{2}\, r} = \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{2\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}\,\pi}{4}$$



$$r^2 + r^2 = d^2$$
$$2r^2 = d^2$$
$$\sqrt{2}\, r = d$$
$$\Rightarrow P_\square = 4d = 4(\sqrt{2}\, r) = 4\sqrt{2}\, r$$

9. Which of the following conditions imply that the real number x is rational?

I. $\sqrt{x}$ is rational
II. x^2 and x^3 are rational
III. x^2 and x^4 are rational

(a) I only
(b) I and II only
(c) I and III only
(d) II and III only
(e) I, II, and III

(i) if $\sqrt{x}$ rational, then $\sqrt{x} = \frac{p}{q} \Rightarrow x = \frac{p^2}{q^2}$ which is still rational, $p, q \in \mathbb{Z}$, $q \neq 0$

(ii) if x^2 + x^3 are rational, then $x^2 = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$.

$\Rightarrow x = \pm\sqrt{\frac{p}{q}} \Rightarrow x^3 = \frac{\pm p\sqrt{p}}{q\sqrt{q}}$ since x^3 rational, then $\frac{\sqrt{p}}{\sqrt{q}}$ also rational, i.e. x rational

(iii) let $x = \sqrt{2}$, $x^2 = 2$, $x^4 = 4$ but x not rational

10. If the roots of $x^2 + bx + c = 0$ are π and $\sqrt{2}$, then $b =$

(a) $\pi\sqrt{2}$
(b) $2\pi\sqrt{2}$
(c) $2(\pi + \sqrt{2})$
(d) $\pi + \sqrt{2}$
(e) $-(\pi + \sqrt{2})$

$x^2 + bx + c = 0$

roots $= \pi$ and $\sqrt{2} \Leftrightarrow$ factors are $(x - \pi)$ and $(x - \sqrt{2})$

$\Rightarrow (x-\pi)(x-\sqrt{2}) = 0$

$x^2 - \pi x - \sqrt{2}x + \sqrt{2}\pi = 0$

$x^2 + (-\pi - \sqrt{2})x + (\sqrt{2}\pi) = 0$

$\Rightarrow b = -\pi - \sqrt{2} = -(\pi + \sqrt{2})$

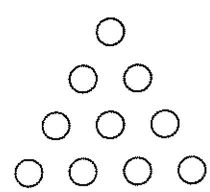
11. One year, there were exactly four Fridays and four Mondays in July. What day of the week was July 20?

(a) Sunday
(b) Monday
(c) Wednesday
(d) Thursday
(e) Saturday

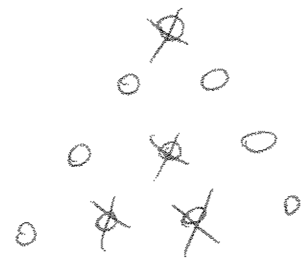
By trial

M	T	W	Th	F	Sa	Sun
	1	2	3	4	5	6
7					12	13
14					19	20
21					26	27
28	29	30	31			

12. What is the smallest number of circles that must be removed from the figure so that no three remaining circles form an equilateral triangle?



(a) 1 (b) 2 (c) 3 (d) 4 (e) 5



13. How many four-digit numbers are there with all odd digits?

(*a*) 625 (*b*) 5! (*c*) 5000 (*d*) 5001 (*e*) 20

can use digits 1, 3, 5, 7 or 9

So we have 5 choices for

each of 4 digits

$\Rightarrow 5(5)(5)(5) = 25(25) = 625$

14. Which of the following numbers is a factor of 68,574,961?

(*a*) 3 (*b*) 9 (*c*) 7 (*d*) 2 (*e*) 11

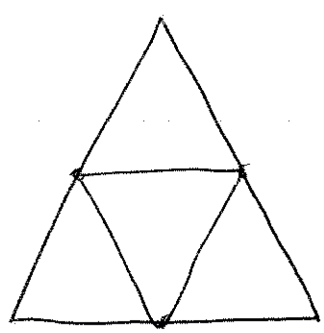
not divisible by 2
nor 3 ⇒ not divisible by 9

check 7 or 11
(or know the divisibility tests
for 7 and 11)

$$
\begin{array}{r}
9796423 \\
7\,\overline{)\,68574961} \\
\underline{63} \\
55 \\
\underline{49} \\
67 \\
\underline{63} \\
44 \\
\underline{42} \\
29 \\
\underline{28} \\
16 \\
\underline{14} \\
21
\end{array}
$$

15. An equilateral triangle is divided into more than one smaller equilateral triangles. What is the smallest possible number of such triangles?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 9



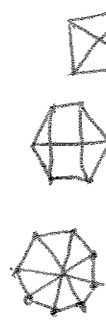
Create more equilateral Δs by connecting midpts.

16. Suppose 50 cities are connected by roads in such a way that three roads lead in and out of each city. How many roads are there total?

(*a*) 50 (*b*) 75 (*c*) 100 (*d*) 125 (*e*) 150

We can only have 3 roads in + out of every city if there is an even # of cities.

n	# cities	# roads
1	2	— NA
2	4	6
3	6	9
4	8	12
⋮	⋮	⋮
n	2n	3n
⋮	⋮	⋮
25	50	75



you can notice this pattern.

OR you can notice that # roads = # cities multiplied by 3 roads each, but we have to divide that in half since we counted each road twice

$\frac{3(50)}{2} = 75$

17. Solve $\log(5x) + \log(x-1) = 2$.

(*a*) 3 (*b*) 5 (*c*) -4 (*d*) 3 and 5 (*e*) 5 and -4

$\log[(5x)(x-1)] = 2$

$10^{\log[5x(x-1)]} = 10^2$

$5x(x-1) = 100$

$5x^2 - 5x - 100 = 0$

$5(x^2 - x - 20) = 0$

$5(x-5)(x+4) = 0$

$x = 5$ or $\cancel{x = -4}$ throw away

domain

$5x > 0$

$\Leftrightarrow x > 0$

and

$x - 1 > 0$

$x > 1$

$\Rightarrow x > 1$

18. How many three-digit whole numbers have the property that doubling them results in reversing their digits? (For instance 125 does not have this property since 2(125) = 250 which does not equal 521, the number obtained by reversing the digits of 125. Also, 025 is not considered a three-digit number, but rather the two-digit number 25.)

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 6 (*e*) none of the above

Let original # be abc w/ value
$100a + 10b + c$.
Then, we need $200a + 20b + 2c$
$= 100c + 10b + a$
leaves us w/ 4 cases:
① $2a = c$, $2b = b$, $2c = a$ $\Rightarrow a = b = c = 0$
② $2a = c$, $b = 2b + 1$, $a = 2c - 10$
$\Rightarrow b = -1/2$
③ $c = 2a + 1$, $b = 2b - 10$, $a = 2c$
$\Rightarrow c = -1/3$
④ $c = 2a + 1$, $b = 2b - 10 + 1$, $a = 2c - 10$
$\Rightarrow c = 19/4$
$\Rightarrow \exists$ no positive integer solutions.

19. A boy and a girl are sitting on the porch. "I'm a boy," says the child with black hair. "I'm a girl," says the child with red hair. At least one of them is lying. What is the maximum number of statements below that can be true?

(I) The person with red hair is a boy.

(II) The person with red hair is a girl.

(III) The person with black hair is a girl.

(*a*) 0
(*b*) 1
(*c*) 2
(*d*) 3
(*e*) There is not enough information to determine the answer.

we know there's one girl + one boy and one has black hair, the other has red hair, and at least one lies.

(I) if red hair is boy, then red hair lied ⇒ black hair is girl + black hair lied (that works).

(II) if red hair is girl, then red hair tells truth ⇒ black hair tells truth (contradiction)

(III) if black hair is girl, then black hair lied ⇒ red hair lied as well (that works)

20. Mary had a coin purse with fifty coins (which are either pennies, nickles, dimes or quarters) totaling exactly \$1.00. Unfortunately, while counting her change, she dropped one coin. What is the probability that it was a penny?

(a) 50%
(b) 75%
(c) 85%
(d) 90%
(e) There is not enough information to determine the answer.

$0.01p + 0.05n + 0.1d + 0.25q = 1$

$\Leftrightarrow \; p + 5n + 10d + 25q = 100$ ①

$p + n + d + q = 50$ ②

① − ② $\Rightarrow 4n + 9d + 24q = 50$

$\Leftrightarrow n = \dfrac{50 - 9d - 24q}{4}$

From this, we can gather exhaustive list of choices.

	p	n	d	q
A	40	8	2	0
B	45	2	2	1

$P(\text{drop penny in } \boxed{A}) = \frac{40}{50} = \frac{8}{10}$

$P(\text{drop penny in } \boxed{B}) = \frac{45}{50} = \frac{9}{10}$

Since both $\boxed{A}$ + $\boxed{B}$ are equally likely, then $P(\text{drop penny}) = \dfrac{\frac{9}{10} + \frac{8}{10}}{2} = 85\%$

21. Given S T A T E M A T H how many arrangements are there of these blocks?

(a) 10! (b) $\frac{10!}{5!}$ (c) $\frac{10!}{3!}$ (d) $\frac{10!}{12}$ (e) $\frac{10!}{6}$

If all the blocks were different, it would be 10!. But since some are the same, it's

$\dfrac{10!}{3!\,2!} = \dfrac{10!}{(3\cdot 2)(2)} = \dfrac{10!}{12}$

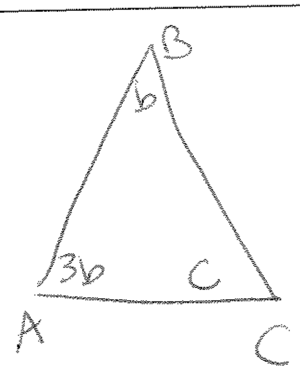
because 3 $\boxed{T}$ and we can't tell difference

because 2 $\boxed{A}$ and we can't tell difference

22. In triangle ABC, the measure of the angle at vertex A is three times

the measure of the angle at vertex B and half the measure of the angle at vertex C. What is the measure of the angle at vertex A?

(*a*) 30° (*b*) 36° (*c*) 54° (*d*) 60° (*e*) 72°



$3b = \frac{1}{2}c$

$6b = c$

and $3b + b + c = 180°$

$3b + b + 6b = 180°$

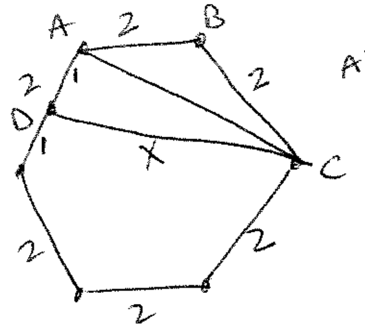
$10b = 180°$

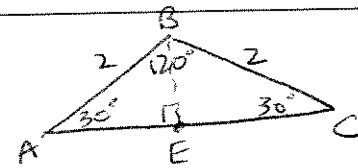
$b = 18°$

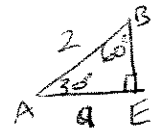
$\Rightarrow A: \quad 3b = 3(18°) = 54°$

23. A park has the shape of a regular hexagon of sides 2 km each. Allice walks a distance of 5km around the perimeter. What is the direct distance between the start point and the end point?

(*a*) $\sqrt{13}$ (*b*) $\sqrt{14}$ (*c*) $\sqrt{15}$ (*d*) $\sqrt{16}$ (*e*) $\sqrt{17}$







We know vertex angle for regular hexagon

$$= \frac{180°(6-2)}{6} = 30(4) = 120°$$

$$\cos 30° = \frac{a}{2}$$

$$\Rightarrow a = 2\cos 30°$$

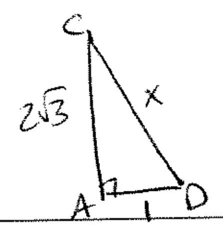
$$a = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$\Rightarrow AC = 2\sqrt{3}$$

$m\angle BAC = 30°$

$m\angle BAC + m\angle CAD = 120°$

$\Rightarrow m\angle CAD = 90°$



$$1^2 + (2\sqrt{3})^2 = x^2$$

$$1 + 12 = x^2 \iff x = \sqrt{13}$$

24. A nanobillion is

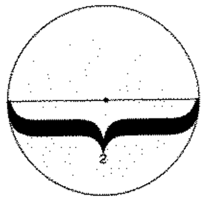
(*a*) .01 (*b*) 0.1 (*c*) 1.0 (*d*) 10 (*e*) 100

"nano" means one billionth

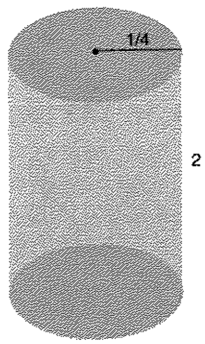
⇒ nanobillion means one billionth of a billion

= 1

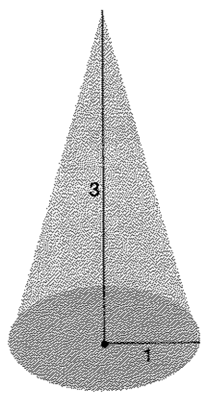
25. Which of the following is **not** equal to π?

(*a*) 

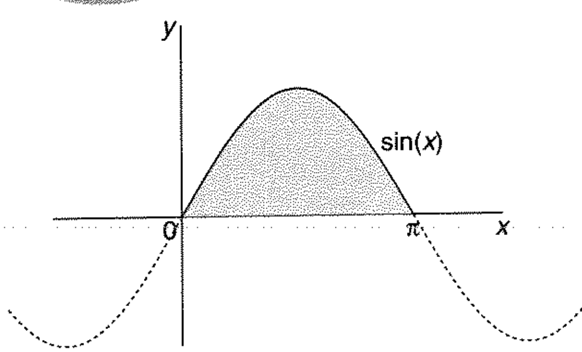
area of this circle

(*b*) 

surface area of the label for this cylinder

(*c*) 

volume of this cone

(*d*) 

shaded area under this curve

(*e*) all of the above are numerically equal to π

(a) $A = \pi(1^2) = \pi$

(b) $SA = 2\pi r h = 2\pi\left(\frac{1}{4}\right)2 = \pi$

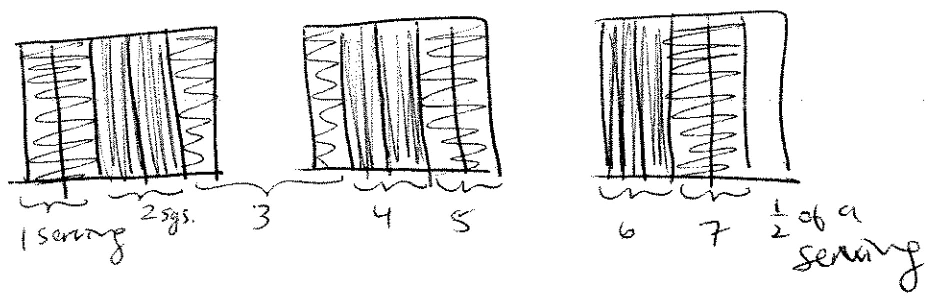
(c) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(1^2)(3) = \pi$

(d) $A = \int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi}$

$= -(-1-1) = 2 \neq \pi$

26. I have three cakes, each divided into 5 equal pieces. A serving is $\frac{2}{5}$ of a cake. How many servings do I have altogether?

(a) $\frac{2}{15}$ (b) $1\frac{1}{5}$ (c) $3\frac{2}{5}$ (d) $7\frac{1}{5}$ (e) $7\frac{1}{2}$

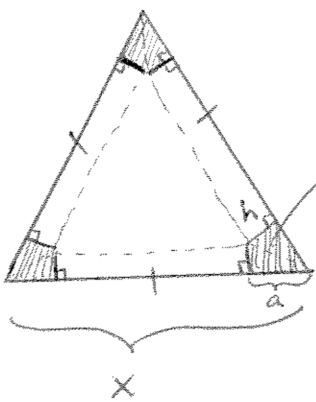


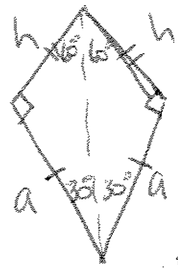
or $3 \div \frac{2}{5} = 3 \cdot \frac{5}{2} = \frac{15}{2} = 7\frac{1}{2}$

27. Given an equilateral triangular piece of cardboard, create an open box (i.e., without a lid) by cutting the same shape from each corner

and folding up the flaps. What is the height of the box of maximal volume? (Assume length of the leg of original cardboard piece is x.)

(a) $\frac{x}{6}$ (b) $\frac{\sqrt{3}x}{18}$ (c) $\frac{x}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{9}x$ (e) $\frac{1}{3}x$





x is constant

h = height of box

$\tan 30° = \frac{h}{a} \Rightarrow a = h\cot 30°$

$a = \sqrt{3}h$

maximize Volume

$V = Ah = \frac{\sqrt{3}}{4}(x - 2\sqrt{3}h)^2 h$

max? min

A = area of base

Area of equilateral Δ =

$\frac{1}{2}(x-2a)\left(\frac{\sqrt{3}}{2}\right)(x-2a)$

$= \frac{\sqrt{3}}{4}(x-2a)^2$

$V = \frac{\sqrt{3}}{4}(x^2 - 4\sqrt{3}xh + 12h^2)h$

$= \frac{\sqrt{3}}{4}(x^2h - 4\sqrt{3}xh^2 + 12h^3)$

$\frac{dV}{dh} = \frac{\sqrt{3}}{4}(x^2 - 8\sqrt{3}xh + 36h^2) = 0$

$36h^2 - 8\sqrt{3}xh + x^2 = 0$

$h = \frac{8\sqrt{3}x \pm \sqrt{192x^2 - 4(36)(x^2)}}{2(36)}$

$h = \frac{8\sqrt{3}x \pm \sqrt{48x^2}}{72} = \frac{8\sqrt{3}x \pm 4\sqrt{3}x}{72}$

$h = \frac{12\sqrt{3}x}{72}$ or $\frac{4\sqrt{3}x}{72}$

$h = \frac{\sqrt{3}x}{6}$ or $\frac{\sqrt{3}x}{18}$

and $V\left(\frac{\sqrt{3}x}{6}\right) = 0$

but $V\left(\frac{\sqrt{3}x}{18}\right) = x^2/54$ = max volume

28. Amaliea is putting her stack of pennies into rolls, keeping out the shiny ones. She notices that every other penny she picks up is dull and every third one is discolored and every fourth one is nicked or bent. How many pennies will she have to roll up if she ends up with fifty shiny pennies?

(a) 50 (b) 100 (c) 120 (d) 150 (e) 160

x = every other one dull d = every third one discolored

$\underbrace{\_\ x\ d\ x\ \_\ xd}\ \underbrace{\_\ x\ d\ x\ \_\ xd}\ \_$

$\underbrace{x\ d\ x\ \_\ xd}\ \_\ x\ d$

The last fact that every 4th penny is nicked doesn't rule out any more pennies.

⇒ Basically, for every 6 pennies, we only keep out 2 shiny pennies, and we roll up 4 pennies.

⇒ for 50 shiny pennies we have 25 groups of 6 total ⇒ 25(4) = 100 rolled up

29. In this expression $ax + by + c = d$, which constants and coefficients determine the y-intercept?

(a) a, b and c
(b) b, c and d
(c) a, b and d
(d) a, c and d
(e) a, b,c and d

$$ax + by + c = d$$
$$by = -ax - c + d$$
$$y = \frac{-a}{b}x + \frac{d-c}{b}$$

⇒ y-intercept is $\left(0, \frac{d-c}{b}\right)$

30. Solve for x.

$$8^{2x} = 2^x \left(\frac{64^6}{2}\right)$$

(a) 5 (b) 6 (c) 7 (d) 10 (e) 11

$$8^{2x} = \frac{2^x((2^6)^6)}{2}$$

$$(2^3)^{2x} = \frac{2^x 2^{36}}{2}$$

$$2^{6x} = 2^{x+36-1}$$

$$\Rightarrow \quad 6x = x + 35$$

$$5x = 35$$

$$x = 7$$