Solutions
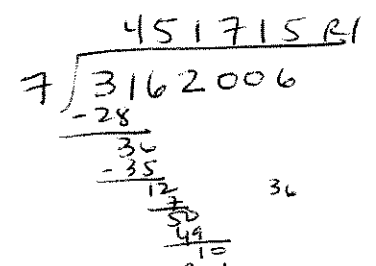## UTAH STATE MATH CONTEST
MARCH 16, 2006
GRADES 7-9

1. The date is 03/16/2006. How many of the first ten counting numbers are factors of 3,162,006?

   a. 3   b. 4   (c.) 5   d. 6   e. 7

   ①②③ 4, 5, ⑥7, 8, ⑨, 10

   $$7\overline{)3162006}$$ = 451715 R1
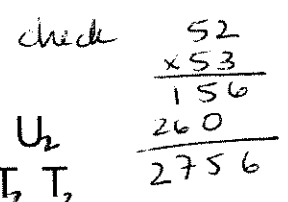
   -28
   36
   -35
   12
   36

2. Paul opens a book at random and notices that the product of the two pages facing one another is 2756. The sum is 105. What is the lower page number?

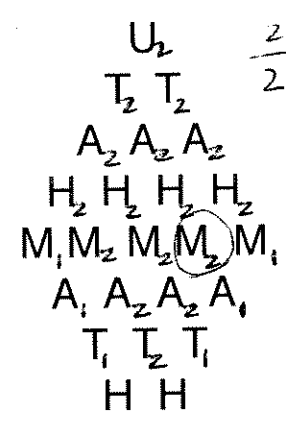   a. 62   b. 51   c. 104   d. 53   (e.) 52

   $xy = 2756$
   $x + y = 105$

   last digit of $x+y$ must be 2+3 so they multiply to be 6 + add to be 5

   check
   52
   ×53
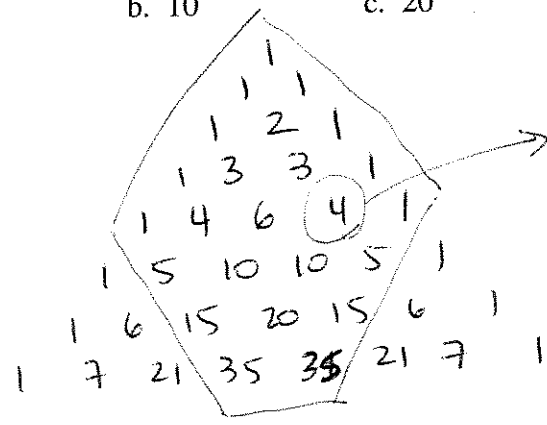   156
   260
   2756

3. Given the array on the right and the rules below, how many different paths can you take to spell UTAHMATH?

   - Begin at the top.
   - Move only down.
   - For each move, go to one of the one or two letters directly below the letter you are on.

   Use Pascal's triangle here. I wrote # of choices next to each spot.

   U_2
   $T_2\ T_2$
   $A_2\ A_2\ A_2$
   $H_2\ H_2\ H_2\ H_2$
   $M_1\ M_2\ M_2\ (M_2)\ M_1$
   $A_1\ A_2\ A_2\ A_1$
   $T_1\ T_2\ T_1$
   $H\ H$

   a. 2   b. 10   c. 20   d. 35   (e.) 70

   1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 (4) 1
   1 5 10 10 5 1
   1 6 15 20 15 6 1
   1 7 21 35 35 21 7 1

   This corresponds to the circled M above. It means we have 4 ways to get to M. ⇒ There are 35 ways to get to bottom right H + another 35 ways to get to bottom left H.

4. Convert $1.2\overline{6}$ to a fraction

    a. $1\dfrac{1}{3}$
    b. $1\dfrac{26}{99}$
    c. $1\dfrac{13}{50}$
    d. $1\dfrac{5}{33}$
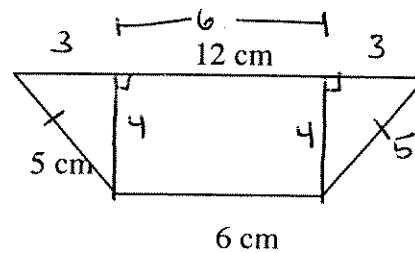   (e.) $1\dfrac{4}{15}$

$$n = 1.2\overline{6}$$

$$1000n = 1266.\overline{6}$$
$$-100n = 126.\overline{6}$$
$$\overline{\qquad\qquad\qquad}$$
$$900n = 1140$$

$$n = \frac{1140}{900} = \frac{114}{90} = \frac{57}{45} = 1\frac{12}{45} = 1\frac{4}{15}$$

5. Find the area of this isosceles trapezoid:

$$A = \tfrac{1}{2}(12+6)\,4 = 2(18)$$
$$= 36 \text{ cm}^2$$

    a. 28 cm²
    (b.) 36 cm²
    c. 45cm²
    d. 48 cm²
    e. 60 cm²

6. Given a triangle, $\triangle ABC$ such that angles A, B, C, measure respectively 50°, 60° and 70°, which side of the triangle is the longest?

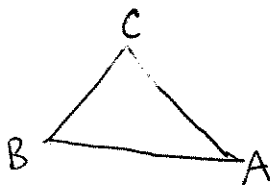   (a.) $\overline{AB}$
    b. $\overline{AC}$
    c. $\overline{BC}$
    d. All sides are the same length
    e. There is not enough information.

The longest leg is opposite largest angle.

7. Students in Ms. Ramirez's math class scored the following totals on a test: 85, 84, 69, 91, 80, 77, 92, 80, 76, 96. What was the median score?
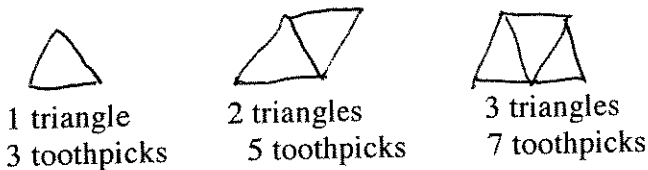
    a. 85   b. 84   (c.) 82   d. 80   e. 83

69   76   77   80   80   84   85   91   92   96

middle scores

Take average (mean) of middle scores.

$$\frac{80+84}{2} = 82$$

8. Aziz is putting triangles together using toothpicks. If he follows this pattern, how many toothpicks will it take to make fifty attached triangles?
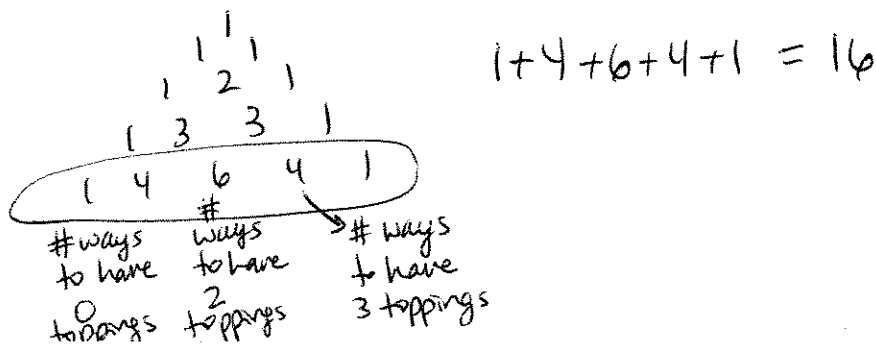
1 triangle
3 toothpicks

2 triangles
5 toothpicks

3 triangles
7 toothpicks

    (a.) 101   b. 100   c. 99   d. 53   e. 51

n	# toothpicks	
1	3	$= 2(1)+1$
2	5	$= 2(2)+1$
3	7	$= 2(3)+1$
⋮		
n	$2n+1$	

So if $n = 50$,

\# toothpicks $= 2(50)+1$
$= 101$

9. At the pizza parlor, they offer four possible toppings. How many different pizzas can be ordered assuming that you may have from 0 to 4 toppings?

    a. 5   b. 8   c. 10   (d.) 16   e. 32

We can either have each topping or not. So, for every topping, we have 2 choices. $\Rightarrow 2^4 = 16$ choices for pizza

Or, we can use Pascal's triangle

1
1   1
1   2   1
1   3   3   1
1   4   6   4   1

$1+4+6+4+1 = 16$

\# ways to have 0 toppings   \# ways to have 2 toppings   \# ways to have 3 toppings

10. A student takes a 30-question math test where they got n questions correct, m questions incorrect and they left k questions blank. If correct answers are worth 4 points, incorrect answers are worth -1 points and blank answers worth zero points, write an expression to describe the student's total score.

a. n-4m+k   b. n+m+k=30   c. n-m   (d) 4n-m   e. n+m+k

$4n$ = pts for correct questions

$-m$ = pts for incorrect questions

0 = pts for blank answers

11. How many ways can you make change for a quarter using pennies, nickels and dimes?

a. 2   b. 3   c. 9   d. 10   (e.) 12

P	25	0	20	10	15	5	10	0	5	0	15	5
n	0	1	1	1	2	2	3	3	4	5	0	0
d	0	2	0	1	0	1	0	1	0	0	1	2

12. Suppose a bag contains the three letters of the word "mom." If you take one letter out at a time and line them up from left to right as you pick them up, what is the probability that you will spell the word "mom?"

a. 1/6   b. 1/4   (c.) 1/3   d. 1/2   e. 1

$$\frac{2 \text{ ways to get mom}}{6 \text{ ways to arrange three letters}} = \frac{1}{3}$$

13. Today is March 16th which is a Thursday. On what day of the week will August 2nd fall?

a. Monday   (b.) Wednesday   c. Friday   d. Sunday   e. None of the previous choices.

Thursdays ⇒ 3/23, 3/30, 4/6, 4/13, 4/20, 4/27, 5/4, 5/11, 5/18, 5/25, 6/1, 6/8, 6/15, 6/22, 6/29, 7/6, 7/13, 7/20, 7/27, 8/3

14. 450 students are surveyed as they enter the cafeteria about which subjects they liked. Every other student liked math, every third student liked English and every fifth student liked Art. How many students did not like math nor art nor English?

a. 120   b. 105   c. 315   d. 135   e. 150

We need LCM for 2, 3 + 5 which is 30.

① 2 3 4 5 6 ⑦ 8 9 10
⑪ 12 ⑬ 14 15 16 ⑰ 18 ⑲ 20
21 22 ㉓ 24 25 26 27 28 ㉙ 30
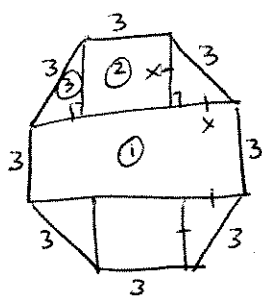
Cross out all student #s that liked one (or more) subjects. It leaves 8 #s out of every group of 30 not included.

$$\Rightarrow \frac{450}{30} = 15 \text{ and } 15(8) = 120$$

15. What is the area of a regular octagon with sides 3 in?

a. $(15+12\sqrt{3})\,\text{in}^2$ b. $(15+15\sqrt{3})\,\text{in}^2$ c. $(13.5+18\sqrt{2})\,\text{in}^2$ d. $(18+18\sqrt{2})\,\text{in}^2$ e. $(27+6\sqrt{2})\,\text{in}^2$

$$x^2 + x^2 = 3^2$$
$$2x^2 = 9$$
$$x^2 = \frac{9}{2}$$
$$x = \frac{3}{\sqrt{2}}$$

$$A = 3\left(3 + 2\left(\frac{3}{\sqrt{2}}\right)\right) + 2\left[3\left(\frac{3}{\sqrt{2}}\right)\right] + 4\left[\frac{1}{2}\left(\frac{9}{2}\right)\right]$$

$$= 3\left(\frac{3\sqrt{2}+6}{\sqrt{2}}\right) + \frac{18}{\sqrt{2}} + 9$$

$$= \frac{9\sqrt{2}+36}{\sqrt{2}} + 9 = \frac{18+36\sqrt{2}}{2} + 9$$

$$= 9 + 18\sqrt{2} + 9$$

$$= 18 + 18\sqrt{2}$$

16. What is the least common multiple of the first ten counting numbers?

a. 1   b. 5040   c. 2520   d. 840   e. 1260

$$1 \quad 2 \quad 3 \quad 2^2 \quad 5 \quad 2\cdot3 \quad 7 \quad 2^3 \quad 3^2 \quad 2\cdot5$$

$$LCM = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 40(63) = 2520$$
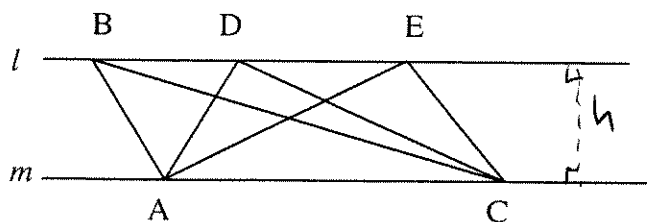
17. What is the greatest common factor of 8! and 4^3?

a. 8!   b. 2^6 c. 2^4 d. 2^8 e. 4

$$8! = 2^3 \cdot 7 \cdot 2\cdot3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \cdot 1 = 2^7 \cdot 3^2 \cdot 5 \cdot 7$$

$$4^3 = (2^2)^3 = 2^6$$

20. Given that the lines l and m are parallel, which of the three triangles has the greatest area, $\triangle ABC$, $\triangle ADC$, $\triangle AEC$?

a. $\triangle ABC$ b. $\triangle ADC$ c. $\triangle AEC$ (d.) The areas are all the same.   e. There is not enough information.

All $\triangle$s have same base $\overline{AC}$ and height h.

18. IF three fair dice are tossed and the product of the numbers that appears is even, what is the probability that the sum of the numbers is also even?

a. 1/2   b. 3/4   c. 5/8   d. 3/7   (e.) 4/7

The ways we can throw 3 dice w/ regard to odd + even is EEE, EED, EOE, OEE, EOO, OEO, OOE, OOO. Out of those, only OOO has an odd product. So we have 7 ways to have an even product. And only EEE, EOO, OEO, OOE will have an even sum.
$\Rightarrow$ 4/7

19. A store has a five-day sale where all merchandise is discounted by 1/3 on the first day. Beginning on the second day and each day thereafter, they take an additional 10% off the previous day's price. What will you pay for a $120 item on the third day? (Round your answer to the nearest whole dollar.)
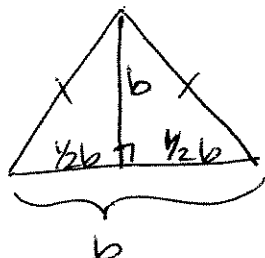
a. $58   b. $55   (c.) $65   d. $64   e. $56

1st day $\Rightarrow$ pay 120\left(\frac{2}{3}\right) = \$80$

2nd day $\Rightarrow$ pay $0.9(80) = \$72$

3rd day $\Rightarrow$ pay $0.9(72) = \$64.80 \approx \$65$

21. If the altitude (height) of an isosceles triangle equals the base and each is numerically equal to the area, what is the area?

    a. 2 units²   b. 1 units²   c. 8 units²   d. 4 units²   e. 16 units²

$$A = \tfrac{1}{2}(b \cdot b) = \frac{b^2}{2} \quad \text{but} \quad A = b \ (given)$$

$$\Rightarrow \quad \frac{b^2}{2} = b$$

$$b^2 = 2b$$

$$b^2 - 2b = 0$$

$$b(b-2) = 0$$

$$b = 2$$

22. If $\dfrac{x}{y} = \dfrac{3}{2}$ and $\dfrac{y}{z} = \dfrac{2}{7}$, then what does $\dfrac{x}{z}$ equal?

    a. $\dfrac{4}{21}$ b. $\dfrac{7}{3}$ c. $\dfrac{21}{4}$ d. $\dfrac{5}{14}$ e. $\dfrac{3}{7}$

$$\frac{y}{z} = \frac{2}{7} \qquad y = \frac{2}{7}z$$

$$\frac{x}{y} = \frac{3}{2} \iff \frac{x}{\frac{2z}{7}} = \frac{3}{2} \iff 14x = 6z$$

$$\frac{x}{z} = \frac{6}{14} = \frac{3}{7}$$

23. A reservoir has vertical sides measuring 20 meters and a rectangular base that measures 30 meters by 40 meters. At the beginning of the summer the reservoir was filled to capacity. At the end of the summer the water depth was 4 meters. How much water was used?
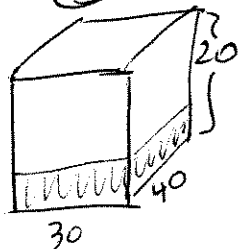
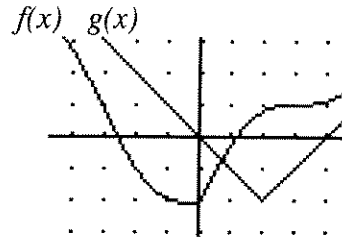   a. 19,200 m³   b. 4,800 m³   c. 28,800 m³   d. 4,640 m³   e. 24,000 m³

Volume of empty space.

$$V = 30(40)(16) = 1200(16) = 19{,}200 \ m^3$$

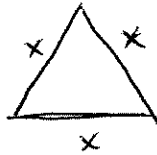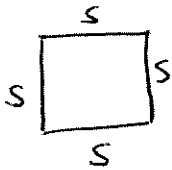**24.** Given $f(x)$ and $g(x)$ as pictured at the right, determine $f(-1)+g(3)$

$f(x)$ $g(x)$

a. 0   b. 3   c. -3   d. -2   e. 2

$f(-1) = -2$ $g(3) = -1$

$f(-1) + g(3) = -2 + -1 = -3$

25. If the perimeter of an equilateral triangle with side x is equal to the perimeter of a square with sides s, what does x equal in terms of s?

a. $x = \sqrt{s}$ b. $x = \dfrac{9s}{16}$ c. $x = \dfrac{s}{12}$ d. $x = s-1$ e. $x = \dfrac{4s}{3}$

$3x = 4s$

$x = \dfrac{4}{3}s$

26. Suppose you have a standard deck of 52 playing cards. Find the probability of drawing either a queen or a club.

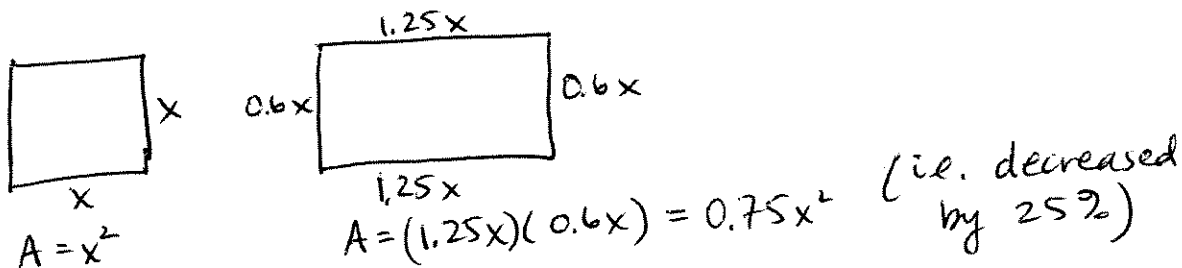a. 4/13   b. 17/52   c. 1/26   d. 9/26   e. 9/13

$P(\text{Queen or club}) = P(\text{Queen}) + P(\text{club}) - P(Q \wedge C)$

$= \dfrac{4}{52} + \dfrac{13}{52} - \dfrac{1}{52}$
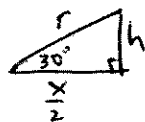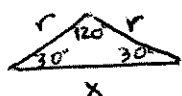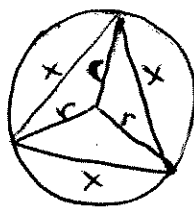
$= \dfrac{16}{52} = \dfrac{4}{13}$

27. Two opposite sides of a square are increased by 25% and the other two are decreased by 40%. What is the percent decrease in the area of the resulting rectangle?

    a. 2.25%   b. 15%   c. 25%   d. 40%   e. 75%

$$1.25x$$
$$x \qquad 0.6x \qquad 0.6x$$
$$x$$
$$1.25x$$
$$A = x^2 \qquad A = (1.25x)(0.6x) = 0.75x^2 \quad (\text{i.e. decreased by 25\%})$$

28. An equilateral triangle is inscribed in a circle. What is the ratio of the area of the triangle to the area of the circle?

e. There is not enough information.

    a. $\dfrac{\sqrt{3}}{\pi}$ b. $\dfrac{2}{\pi}$ c. $\dfrac{2\sqrt{3}}{\pi}$ d. $\dfrac{3\sqrt{3}}{4\pi}$

$$\cos 30° = \frac{x/2}{r} \Rightarrow x = \sqrt{3}\,r$$
$$\sin 30° = \frac{h}{r} \Rightarrow h = \frac{1}{2}r$$
$$A_\triangle = 3\left(\frac{1}{2}xh\right) = \frac{3}{2}\left(\sqrt{3}r\right)\left(\frac{r}{2}\right) = \frac{3\sqrt{3}}{4}r^2$$
$$A_\circ = \pi r^2$$
$$\frac{A_\triangle}{A_\circ} = \frac{\frac{3\sqrt{3}}{4}r^2}{\pi r^2} = \boxed{\frac{3\sqrt{3}}{4\pi}}$$

29. Suppose it takes h minutes to fill a bath tub using the hot water faucet and c minutes to fill the same tub using the cold water faucet. Starting with an empty tub, the hot water faucet is turned on and then after 1 minute, the cold water faucet is also turned on. How long will it take to fill the tub?

    a. $\dfrac{h+(c-1)}{2}$ b. $\dfrac{h(c+1)}{2}$ c. $\dfrac{h(c+1)}{h+c}$ d. $\dfrac{hc}{c+h}$ e. $\dfrac{hc}{h+c}-1$

After 1 minute, there's only $1-\frac{1}{h}$ of the job to finish.

And $\frac{1}{h}+\frac{1}{c}=\frac{1}{t} \Rightarrow t = \frac{hc}{h+c} =$ time it takes to do entire job together

$\Rightarrow$ total time $= 1 + \underbrace{\left(1-\frac{1}{h}\right)\left(\frac{hc}{h+c}\right)}_{\substack{\text{rest of job} \\ \text{together}}} = \frac{h+c+(h-1)c}{h+c}$

$\underbrace{1\,\text{min}}$

$= \frac{h+hc}{h+c} = \frac{h(1+c)}{h+c}$

30. Find x if $x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 2$

a. $\pm\sqrt{2}$ b. $2 \pm \sqrt{2}$ c. $\dfrac{1}{2}$ d. $2 \pm 2i\sqrt{2}$ (e.) $2 - \sqrt{2}$

$$x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 2$$

$$\left(\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}\right)^2 = (2 - x)^2$$

$$x + \sqrt{x + \sqrt{x + \ldots}} = 4 - 4x + x^2 \qquad \left(\text{but } x + \sqrt{x + \sqrt{x + \ldots}} = 2\right)$$

$$2 = 4 - 4x + x^2$$

$$0 = x^2 - 4x + 2$$

$$x = \frac{4 \pm \sqrt{16 - 4(2)}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

But $\qquad 2 + \sqrt{2} + \sqrt{2 + \sqrt{2} + \sqrt{2 + \sqrt{2} + \ldots}} = 2$

means $\qquad \sqrt{2} + \sqrt{2 + \sqrt{2} + \sqrt{2 + \sqrt{2} + \ldots}} = 0$

i.e. $\qquad \sqrt{2} + \text{something positive} = 0 \qquad$ which can't happen

$\Rightarrow \quad 2 - \sqrt{2}$ is only answer. ☺