PhD qualifying exam in statistics–2015, Summer

You need to collect at least 60 points to pass the exam. You will get 10 points for the correct solution of each question.

1. Let $X_1, X_2, \ldots, X_n$ be independent, random variables with density functions

$$h(t;\theta_i) = \begin{cases} \frac{1}{\theta_i} e^{-t/\theta_i}, & \text{if } 0 \le t < \infty \\ 0, & \text{if } -\infty < t < 0. \end{cases}$$

We wish to test $H_0 : \theta_1 = \theta_2 = \ldots \theta_n$ against the alternative that H_0 is not true.
(a) Compute λ, the generalized likelihood ratio and explain the application of the generalized likelihood test.
(b) Provide a large sample approximation for $-2 \log \lambda$ under H_0.

2. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 \le t \le \theta \\ 0, & \text{if } t \notin [0, \theta]. \end{cases}$$

We wish to test $H_0 : \theta = 1$ against the alternative $H_A : \theta < 1$. We reject H_0 if $X_{n,n} = \max_{1 \le i \le n} X_i \le 1 - 2/n$.
(a) Compute the significance level.
(b) Compute the power function.

3. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with density function

$$h(t;\sigma) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(t-1)^2/2\sigma^2}.$$

We wish to test $H_0 : \sigma = \sigma_0$ against $H_A : \sigma \ne \sigma_0$.
(a) Find a test, using the generalized likelihood ratio λ.
(b) Provide a large sample approximation for $-2 \log \lambda$ under H_0.

4. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with density function

$$h(t;\mu) = \frac{1}{(2\pi)^{1/2}} e^{-(t-\mu)^2/2}.$$

We wish to test $H_0 : \mu \le \mu_0$ against $H_A : \mu > \mu_0$. Find the uniformly most powerful test of size α.

5. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed normal $N(\mu_1, \sigma^2)$ random variables. Let $Y_1, Y_2, \ldots, Y_m$ be independent identically distributed normal $N(\mu_2, \sigma^2)$ random variables. The two samples are independent. The parameters μ_1, μ_2, σ^2 are unknown. We wish to test $H_0 : \mu_1 = \mu_2$. Show that the two sample t–test and the likelihood ratio test are equivalent.

6. Let X_1, X_2 be independent and identically distributed exponential(λ) random variables, i.e. the common density is

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{\lambda}\exp(-t/\lambda), & \text{if } t \geq 0. \end{cases}$$

Find the joint density function of

$$Y_1 = X_1 + X_2 \quad \text{and} \quad Y_2 = \frac{X_1}{X_1 + X_2}.$$

7. We assume that the number of people entering a store follows a Poisson distribution with parameter λ. Every customer spends money in the store, independently of each other according to a uniform distribution on $[10, 100]$. We observe X, the total amount of money spent in the store on a given day.
(a) Estimate λ using the method of moments.
(b) Let assume that we have the stores income $X_1, X_2, \ldots, X_n$ for n days and $X_1, X_2, \ldots, X_n$ are independent and distributed as X. What would be the moment estimator for λ based on $X_1, X_2, \ldots, X_n$?

8. Let X and Y be independent random variables with densities f and g, where

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{2}\exp(-t/2), & \text{if } t \geq 0. \end{cases}$$

and

$$g(t) = \begin{cases} 0, & \text{if } t \notin [-1, 2] \\ \frac{1}{3}, & \text{if } t \in [-1, 2]. \end{cases}$$

Compute the density of $X + Y$.

9. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with density function

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{\lambda}\exp(-t/\lambda), & \text{if } t \geq 0. \end{cases}$$

Construct an equal tail $1-\alpha$ confidence interval for λ. Compute the expected value of the length of the confidence interval.

10. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with density function

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{\lambda}\exp(-t/\lambda), & \text{if } t \geq 0. \end{cases}$$

Find the uniformly minimum variance unbiased estimator for $\theta = 1/\lambda$.

11. Let $U_1, U_2, \ldots, U_n$ be independent and identically distributed random variables, uniform on $[0, 1]$. Let $U_{1,n} \leq U_{2,n}, \ldots, \leq U_{n,n}$ denote the order statistics.
(a) Compute the limit distribition of $Z = n(U_{3,n} - U_{2,n})$.
(b) Show that $Z_1 = n(U_{3,n} - U_{2,n})$ and $Z_2 = n(U_{4,n} - U_{3,n})$ are asymptotically independent.

12. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with distribution function F. We wish to test that F is a normal distribution function. Provide at least two different methods on to test for the normality of F.

Discrete Distributions

Bernoulli

$f(x) = p^x(1-p)^{1-x},\ x = 0, 1$

$M(t) = 1 - p + pe^t$

$\mu = p,\ \sigma^2 = p(1-p)$

Binomial
$b(n, p)$

$$f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x},\ x = 0, 1, 2, \ldots, n$$

$M(t) = (1 - p + pe^t)^n$

$\mu = np,\ \sigma^2 = np(1-p)$

Geometric

$f(x) = (1-p)^{x-1}p,\ x = 1, 2, \ldots$

$$M(t) = \frac{pe^t}{1-(1-p)e^t},\ t < -\ln(1-p)$$

$$\mu = \frac{1}{p},\ \sigma^2 = \frac{1-p}{p^2}$$

Hypergeometric

$$f(x) = \frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{r}},\ x \le r,\ x \le n_1,\ r - x \le n_2$$

$$\mu = r\left(\frac{n_1}{n}\right),\ \sigma^2 = r\left(\frac{n_1}{n}\right)\left(\frac{n_2}{n}\right)\left(\frac{n-r}{n-1}\right)$$

Negative Binomial

$$f(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r},\ x = r, r+1, r+2, \ldots$$

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r},\ t < -\ln(1-p)$$

$$\mu = r\left(\frac{1}{p}\right),\ \sigma^2 = \frac{r(1-p)}{p^2}$$

Poisson

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, 2, \ldots$$

$M(t) = e^{\lambda(e^t-1)}$

$\mu = \lambda,\ \sigma^2 = \lambda$

Uniform

$$f(x) = \frac{1}{m},\ x = 1, 2, \ldots, m$$

$$\mu = \frac{m+1}{2},\ \sigma^2 = \frac{m^2-1}{12}$$

Continuous Distributions

Beta

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 < x < 1$$

$$\mu = \frac{\alpha}{\alpha+\beta}$$

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

Chi-Square
$\chi^2(r)$

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2},\ 0 \le x < \infty$$

$$M(t) = \frac{1}{(1-2t)^{r/2}},\ t < \frac{1}{2}$$

$$\mu = r,\ \sigma^2 = 2r$$

Exponential

$$f(x) = \frac{1}{\theta} e^{-x/\theta},\ 0 \le x < \infty$$

$$M(t) = \frac{1}{1-\theta t},\ t < 1/\theta$$

$$\mu = \theta,\ \sigma^2 = \theta^2$$

Gamma

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta},\ 0 \le x < \infty$$

$$M(t) = \frac{1}{(1-\theta t)^\alpha},\ t < 1/\theta$$

$$\mu = \alpha\theta,\ \sigma^2 = \alpha\theta^2$$

Normal
$N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]},\ -\infty < x < \infty$$

$$M(t) = e^{\mu t + \sigma^2 t^2/2}$$

$$E(X) = \mu,\ \mathrm{Var}(X) = \sigma^2$$

Uniform
$U(a, b)$

$$f(x) = \frac{1}{b-a},\ a \le x \le b$$

$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)},\ t \ne 0;\ M(0) = 1$$

$$\mu = \frac{a+b}{2},\ \sigma^2 = \frac{(b-a)^2}{12}$$