DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
REAL AND COMPLEX ANALYSIS PRELIMINARY EXAMINATION

Instructions. You must attempt at least 7 problems from this exam. To pass, you must demonstrate mastery of both real and complex analysis. Getting 3 problems correct on each section is sufficient to do this.

PART A: REAL ANALYSIS

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $1 \leq s < t \leq \infty$. Show that $L^t(X) \subset L^s(X)$.

Problem 2.
Let H be a Hilbert space, $V \subset H$ a closed subspace, and $F \subset H$ a finite dimensional subspace, with $F \cap V = \{0\}$. show that there is a positive constant $\lambda < 1$ such that $| < u, v > | \leq \lambda \|u\| \|v\|$ for all $u \in F$ and $v \in V$.

Problem 3. Consider the interval $[-1, 1]$ with a finite Borel measure μ (not assumed to be the Lebesgue measure). Let $C([-1, 1])$ be the space of continuous functions endowed with the L^2 norm, and let $T : C([-1, 1]) \to \mathbb{R}$ be the operator $T(f) = f(0)$. Show that T is a bounded operator (with respect to the L^2 norm) if and only if $\mu(\{0\}) > 0$.

Problem 4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose that $A_n \, in \mathcal{M}$, $n = 1, \ldots,$ with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that the set of x belonging to infinitely many A_n has measure zero.

Problem 5. For $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, define $\tau_y f(x) = f(x - y)$ for $x \in \mathbb{R}$. Assume that $\mathbb{R}$ is equipped with the Lebesgue measure. For $f \in L^1(\mathbb{R})$ show that the map $\mathbb{R} \to L^1(\mathbb{R})$ defined by $y \to \tau_y f$ is uniformly continuous.

Part B: Complex Analysis

Problem 1. Let $f(z)$ be an analytic function. Show that the successive derivatives of $f(z)$ at a point can never satisfy $|f^{(n)}(z)| > n!n^n$ for all n.

Problem 2. Let $\wp(z)$ be the Weierstrass $\wp$ function. Prove that

$$\wp(z+u) + \wp(z) + \wp(u) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2.$$

Problem 3. Prove that every 1-1 conformal mapping of a disk onto another is given by a linear transformation.

Problem 4. Evaluation the integral by the method of residue:

$$\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} dx.$$

Problem 5. The Bernoulli numbers B_k are defined in terms of the Laurent series

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}.$$

(a) Find the Laurent series of $\cot \pi z$ in terms of the Bernoulli numbers.
(b) Using (a) and the fact on partial fractions

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - n^2}$$

to prove

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k}$$

for any positive integer k.