Probability Prelim

August 19, 2008

There are 10 problems, of which you should turn in solutions for 6 (your best 6). Each problem is worth 10 points, and 40 points is required for passing.

1. Let $X_1, X_2, \ldots$ be random variables such that the partial sums $S_n := X_1 + X_2 + \cdots + X_n$ determine a martingale. Show that $E[X_i X_j] = 0$ if $i \neq j$.

2. Let $X_2, X_3, \ldots$ be independent random variables such that

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n \log n}, \quad P(X_n = 0) = 1 - \frac{1}{n \log n}.$$

Show that this sequence obeys the weak law but not the strong law of large numbers, in the sense that $n^{-1} \sum_{i=1}^{n} X_i$ converges to 0 in probability but not almost surely.

3. Let $\{X_n,\ n \geq 1\}$ be independent $N(0,1)$ random variables. Show that $P(\limsup |X_n|/\sqrt{\log n} = \sqrt{2}) = 1$.

Hint: $1 - \Phi(x)$ is asymptotic to $\phi(x)/x$ as $x \to \infty$.

4. Let $X_1, X_2, \ldots$ be independent random variables such that

$$X_n = \begin{cases} 1 & \text{with probability } (2n)^{-1}, \\ 0 & \text{with probability } 1 - n^{-1}, \\ -1 & \text{with probability } (2n)^{-1}. \end{cases}$$

Let $Y_1 := X_1$ and for $n \geq 2$

$$Y_n := \begin{cases} X_n & \text{if } Y_{n-1} = 0, \\ nY_{n-1}|X_n| & \text{if } Y_{n-1} \neq 0. \end{cases}$$

Show that Y_n is a martingale with respect to $\mathscr{F}_n := \sigma(Y_1, Y_2, \ldots, Y_n)$. Show that Y_n does not converge almost surely. Does Y_n converge in any way? Why does the martingale convergence theorem not apply?

5. Let X and Y be independent exponential random variables with parameters λ and μ. (For example, $P(X > x) = e^{-\lambda x}$ for $x > 0$.)
(a) Show that $Z := \min(X, Y)$ is independent of the event $\{X < Y\}$.
(b) Find $P(X = Z)$.
(c) Find the distribution of $U := (X - Y)^+ = \max(X - Y, 0)$.
(d) Find the distribution of $V := \max(X, Y) - \min(X, Y)$.

6. A die is thrown 10 times. What is the probability that the sum of the scores in 27?
Hint: Use probability generating functions.

7. Suppose $W := (X, Y)$ is a two-dimensional random variable and QW^T is $N(0, 1)$ for all 2×2 rotation matrices

$$Q := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Prove that X and Y are independent and $N(0, 1)$.

8. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ denote random variables on a common probability space. Suppose X_n converges to 0 in probability as $n \to \infty$, and Y_n converges to Y weakly. Prove that (X_n, Y_n) converges weakly to $(0, Y)$.

9. Construct two random variables X and Y such that:
(a) X is $N(0, 1)$,
(b) Y is $N(0, 1)$, and
(c) (X, Y) is not a Gaussian random vector.

10. Suppose X and Y are independent bounded random variables. Prove in complete detail the following assertions:
(a) $E(XY) = E(X)E(Y)$,
(b) $E(X \mid Y) = E(X)$.