Probability Prelim Exam

January 2019

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let X_n be a sequence of i.i.d. random variables. Assume that $P(X_n \to \infty) > 0$. Prove that $P(X_0 = \infty) = 1$. **Hint:** first prove that $P(X_n \to \infty) = 1$.

2. Let $X_n : \Omega \to \mathbb{R}$ be a sequence of integrable random variables with the same distribution. Prove that $X_n/n \to 0$ almost surely. **Hint:** Borel-Cantelli.

3. Let X and Y be two random variables such that $P(X \leqslant Y) = 1$. Assume also that $P(X \leqslant x) \leqslant P(Y \leqslant x)$ for all $x \in \mathbb{R}$. Prove that $P(X = Y) = 1$.

4. (Pólya's Urn) An urn initially contains r red and b blue balls. A ball is chosen uniformly at random (i.e. with probability $1/(r+b)$ each). If If it comes up red (resp. blue), then it is returned and another red (resp. blue) ball is added to the urn. The process is repeated indefinitely. Let R_n be the number of red balls in the urn after n draws.

   a) Prove that the fraction of red balls $M_n = R_n/(r+b+n)$ has an almost sure limit M_∞ as $n \to \infty$.

   b) Suppose now $r = b = 1$. Prove that $P(R_n = k) = 1/(n+1)$ for $1 \leqslant k \leqslant n+1$. **Hint:** Write down the probability of choosing k red and $n-k$ blue balls in some fixed order.

   c) Still assuming $r = b = 1$, what is the distribution of M_∞? Hint: use part b) to compute $P(M_\infty \leqslant x)$.

5. Let X_n be a sequence of independent random variables with $P(X_n = 1) = 1/n = 1 - P(X_n = 0)$. Prove that $\left(\sum_{i=1}^n X_i - \log n\right)/\sqrt{\log n}$ converges weakly to a standard normal.

6. Fix $\alpha \in (0,1)$. Let X_n be a sequence of independent random variables with $P(X_n = n) = P(X_n = -n) = n^{-\alpha}/2$ and $P(X_n = 0) = 1 - n^{-\alpha}$. Prove that $(X_1 + \cdots + X_n)/n^{(3-\alpha)/2}$ converges weakly to a centered normal.

7. Let M_n be a martingale with $M_0 = 0$ and increments $X_n = M_n - M_{n-1}$ that satisfy $E[X_n^2] < \infty$ for all n. Assume that $(b_n)_{n=1}^\infty$ is an increasing sequence of positive real numbers such that $b_n \to \infty$ and

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}[X_n^2]}{b_n^2} < \infty.$$

Show that $M_n/b_n \to 0$ almost surely as $n \to \infty$. **Hint:** Since the sum of the X_k is a martingale so too should be a weighted sum; prove this idea and use it to produce a new martingale out of the X_k and b_k. Derive convergence properties of this new martingale, and then use Kronecker's Lemma: if a_n is a sequence of numbers such that $\sum_{n=1}^\infty a_n$ exists and is finite and b_n is a sequence of the type above, then

$$\frac{1}{b_n} \sum_{k=1}^{n} a_k b_k \to 0 \text{ as } n \to \infty.$$

8. Let $X_1, X_2, \ldots$ be iid random variables with common pdf

$$f(x) = \begin{cases} 0, & |x| \le 2 \\ \frac{c}{x^2 \log |x|}, & |x| > 2 \end{cases}$$

   (a) Prove that there is a finite constant c which makes f a pdf.

   (b) Let $S_n = X_1 + X_2 + \ldots + X_n$. Prove that $S_n/n \to 0$ in probability as $n \to \infty$. **Hint:** Prove that S_n/n converges to 0 in distribution, then explain why you can upgrade to convergence in probability for free.

   (c) Does $S_n/n \to 0$ almost surely as $n \to \infty$?

9. Let X and Y be two integer-valued random variables defined on the same probability space. The *total variation distance* between X and Y is defined as

$$d_{TV}(X,Y) = 2 \sup_{A \subseteq \mathbb{Z}} |P(X \in A) - P(Y \in A)|.$$

Prove that $\mathbb{P}(X = Y) \le 1 - \frac{1}{2} d_{TV}(X,Y)$.

10. Let $g : \mathbb{R} \to \mathbb{R}$ be bounded and continuous and fix $\lambda > 0$. Use properties of the Poisson distribution to show that

$$\sum_{k=0}^{\infty} g(k/n) \frac{(n\lambda)^k}{k!} e^{-n\lambda} \to g(\lambda) \text{ as } n \to \infty.$$